Instructions: Complete all six problems to get full credit. The exercises are in Chapter 1 of our textbook Rudin’s *Principles of Mathematical Analysis*. This homework is due on Tuesday September 1st, 2009.

1. **(#3 in Rudin)** Prove Proposition 1.15: The axioms of multiplication imply the following statements for all $x, y,$ and $z$ in a field $F$.
   
   (a) If $x \neq 0$ and $xy = xz$ then $y = z$.
   
   (b) If $x \neq 0$ and $xy = x$ then $y = 1$.
   
   (c) If $x \neq 0$ and $xy = 1$ then $y = 1/x$.
   
   (d) If $x \neq 0$ then $1/(1/x) = x$.

   Note that (b) and (c) assert the uniqueness of the multiplicative identity and the multiplicative inverse respectively.

2. **(#5 in Rudin)** Let $A$ be a nonempty set of real numbers which is bounded below. Let $-A$ be the set of all numbers $-x$, where $x \in A$. Prove that
   
   $$\inf A = -\sup(-A).$$

   Where the $\inf A$ is defined to be the greatest lower bound of $A$.

3. **(#8 in Rudin)** Prove that no order can be defined in the complex field that turns it into an ordered field. *Hint: $-1$ is a square.*

4. **(#9 in Rudin)** Suppose $z = a + ib$, $w = u + iv$, and
   
   $$a = \left( \frac{|w| + u}{2} \right)^{1/2}, \quad b = \left( \frac{|w| - u}{2} \right)^{1/2}.$$ 

   Prove that $z^2 = w$ if $v \geq 0$ and that $(\overline{z})^2 = w$ if $v \leq 0$. Conclude that every complex number (with one exception!) has two complex square roots.

5. **(#13 in Rudin)** If $x, y$ are complex, prove that
   
   $$| |x| - |y| | \leq |x - y|.$$ 

6. **(#15 in Rudin)** Under what conditions does equality hold in the Cauchy-Schwarz inequality? Recall that the aforementioned inequality says that if $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are complex
numbers, then

\[ \left| \sum_{j=1}^{n} a_j b_j \right|^2 \leq \sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2. \]