Exercise: Let \( f : [1, \infty) \to \mathbb{R} \) be a decreasing, continuous function such that \( \exists M > 0 \) such that for all \( N > 0 \)

\[
\left| \int_{[1,N]} f(x) \, dx \right| < M
\]

Show that \( \lim_{x \to \infty} xf(x) = 0 \).

Sketch of the proof:
- First we will show that \( f(x) \geq 0 \ \forall \ x \)
- Then we will show that \( \lim_{x \to \infty} f(x) = 0 \)
- Next we will give a “hard” proof that \( \lim_{x \to \infty} xf(x) = 0 \) (by contradiction)
- Then we will give an “easy” proof that \( \lim_{x \to \infty} xf(x) = 0 \) (by cleverness)
- Finally we will crack open a couple beers and call it a night.

Lemma 1. \( f(x) \geq 0 \ \forall x \).

Proof. Assume not. That is, assume \( f(x) < 0 \) for some \( x \). So we have say, \( f(x_0) = -a \) for \( a > 0 \).

But since \( f \) is decreasing we have that \( f(x) \leq -a \ \forall \ x > x_0 \).

Thus we get the following:

\[
\int_1^N f(x) \, dx = \int_1^{x_0} f(x) \, dx + \int_{x_0}^N f(x) \, dx \\
\leq L + \int_{x_0}^N (-a \, dx) \\
= L - a(N - x_0)
\]

So, \( \int_1^N f(x) \, dx \leq L - a(N - x_0) \)

Now if we let \( N \to \infty \) we get that:

\[
\int_1^\infty f(x) \, dx \leq -\infty
\]

But this contradicts that the integral is bounded. Hence our assumption that there is some \( x \) such that \( f(x) < 0 \) is wrong, and therefore \( f(x) \geq 0 \ \forall x \). \( \square \)

Lemma 2. \( \lim_{x \to \infty} f(x) = 0 \)

Proof. Assume not. That is, \( \lim_{x \to \infty} f(x) \neq 0 \) or does not exist. Note that since \( f \) is continuous, decreasing, and bounded from below, we have that the limit does exist (Proposition 6.3.8).

From Lemma 1 we know that \( f(x) \geq 0 \ \forall x \). So assume that \( \lim_{x \to \infty} f(x) = a > 0 \). But since \( f \) is decreasing, we have that \( f(x) > a/2 > 0 \ \forall x \) (as if it were otherwise then the limit would not approach \( a \)). Now let us look at the following integral:
\[
\int_1^N f(x)dx > \int_1^N (a/2)dx = (a/2)(N-1)
\]

Now if we let \( N \to \infty \) we get that:
\[
\int_1^\infty f(x)dx > \infty
\]

But this contradicts that the integral is bounded.

Hence our original assumption that \( \lim_{x \to \infty} f(x) \neq 0 \) was false. And therefore \( \lim_{x \to \infty} f(x) = 0 \). \( \square \)

Now we will give the “hard” proof that \( \lim_{x \to \infty} xf(x) = 0 \):

Proof. We will prove this by contradiction. That is, assume \( \exists \epsilon > 0 \) such that \( \forall N > 0 \ \exists x_n > N \) such that \( |x_n f(x_n)| > \epsilon \).

Note that by passing through a subsequence we can assume that \( (x_n) \) is increasing. And since we have the above inequality for all \( N > 0 \) we can choose \( x_n \geq 2x_{n-1} \) (Just by repeatedly taking \( N \) large enough), and we will define \( x_0 = 1 \).

Thus \( x_n - x_{n-1} \geq \frac{x_n}{2} \).

Since \( f \) is decreasing we have \( \int_{x_{n-1}}^{x_n} f(x)dx \geq f(x_n)(x_n - x_{n-1}) \) \( \forall n \geq 1 \)

And from the Integral test we have:
\[
\int_1^\infty f(x)dx = \sum_{n=1}^{\infty} \int_{x_{n-1}}^{x_n} f(x)dx
\]
\[
\geq \sum_{n=1}^{\infty} f(x_n)(x_n - x_{n-1})
\]
\[
\geq \sum_{n=1}^{\infty} f(x_n) \frac{x_n}{2} \quad \text{Since } x_n - x_{n-1} \geq \frac{x_n}{2}
\]
\[
= \frac{1}{2} \sum_{n=1}^{\infty} f(x_n)x_n
\]
\[
> \frac{1}{2} \sum_{n=1}^{\infty} \epsilon = \infty
\]

So we have \( \int_1^\infty f(x)dx > \infty \), a contradiction.

Thus our original assumption was wrong, and therefore we must have \( \lim_{x \to \infty} xf(x) = 0 \). \( \square \)

Here is an interesting way to prove that \( \lim_{x \to \infty} xf(x) \neq 0 \). Assume \( \lim_{x \to \infty} xf(x) = c > 0 \). Note that this is not quite sufficient for what we wanted to show above, as the limit might not exist, and this does not take care of that.

Since \( f \) is decreasing we can use the integral test and we get that:
\[0 \leq \sum_{n=2}^{\infty} f(n) \leq \int_{1}^{N} f(x)dx\]

Since \(f\) is decreasing we also get the following:

\[
\underbrace{f(2) + f(3) + f(4) + \cdots + f(7)} + \underbrace{f(8) + \cdots + f(16)} + \cdots \\
\geq 2f(4) + 4f(8) + 8f(16) + \cdots + 2^n f(2^{n+1}) + \cdots
\]

So we have:

\[
\int_{1}^{\infty} f(x)dx \geq \sum_{n=2}^{\infty} f(n) \geq \frac{1}{2} \sum_{n=1}^{\infty} 2^{n+1} f(2^n+1)
\]

But \(2^{n+1} f(2^{n+1})\) is \(a_{n+1}\), and from our assumption \(\lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} (2^{n+1} f(2^{n+1})) = c > 0\)

But from the Divergence test, this sum diverges. This is a contradiction since our integral is bounded.

Now we will give Cristina’s clever proof:

**Proof.** Since \(f\) is a decreasing function and since \(f(x) \leq f(t)\) for \(x/2 \leq t \leq x\), we get the following inequalities:

\[
\int_{x/2}^{x} f(t)dt \geq f(x)(x - x/2) = f(x)\frac{x}{2} \geq 0
\]

Let \(x \to \infty\). Since the tail of \(\int_{0}^{\infty} f(x)dx\) goes to zero we have that \(\int_{x/2}^{x} f(t)dt \to 0\) as \(x \to \infty\).

Therefore \(\lim_{x \to \infty} xf(x) = 0\). \(\square\)