Exercise: Let $I$ be a bounded interval, and let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be Riemann integrable functions on $I$. Let $a, b \in \mathbb{R}$, prove that $af + bg$ is Riemann integrable, and that $\int_I (af + bg) = a(\int_I f) + b(\int_I g)$.

We will prove this exercise as follows:
- First we will prove a lemma for linearity and integration of piecewise constant(step) functions.
- Then we will prove that $f + g$ is Riemann integrable, and that $\int_I (f + g) = \int_I f + \int_I g$.
- Next we will prove that $cf$ is Riemann integrable, and that $\int_I cf = c(\int_I f)$ (first for $c \geq 0$, and then for $c < 0$).
- Together, this proves that $af + bg$ is Riemann integrable, and that $\int_I (af + bg) = a(\int_I f) + b(\int_I g)$.

First let us prove the following lemma.

**Lemma 1.** Let $I$ be a bounded interval, $a, b \in \mathbb{R}$, and $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be piecewise constant functions on $I$, then $af + bg$ is also piecewise constant on $I$ and $p.c.\int_I (af + bg) = a(p.c.\int_I f) + b(p.c.\int_I g)$.

**Proof.** First note that we can assume that $f, g$ are piecewise constant with respect to the same partition $\mathcal{P}$ since if they were not, we could take the common refinement.

Since $f, g$ are piecewise constant w.r.t. $\mathcal{P}$ we have that $f(x) = c_J, g(x) = d_J \forall x \in J \in \mathcal{P}$. So $(af + bg)(x) = ac_J + bd_J \forall x \in J \in \mathcal{P}$. Thus $af + bg$ is piecewise constant on $I$.

\[
af + bg = \sum_{J \in \mathcal{P}} (ac_J + bd_J) J \]

Giving us that $a(p.c.\int_I f) + b(p.c.\int_I g) = p.c.\int_I (af + bg)$, as desired.

Now let us prove that $f + g$ is Riemann integrable and that $\int_I (f + g) = \int_I f + \int_I g$.

**Proof.** Since $f, g$ are Riemann integrable, given $\epsilon > 0$ we can find piecewise constant functions $\underline{f}, \overline{f}, \underline{g}, \overline{g}$ such that:

\[
\underline{f}(x) \leq f(x) \leq \overline{f}(x) \quad \forall x \in I
\]

and
\[ g(x) \leq f(x) \leq \bar{g}(x) \quad \forall x \in I \tag{2} \]

and also that:

\[
\int_I f - \epsilon \leq \int_I f \leq \int_I f = \int_I f \leq \int_I f \leq \int_I f + \epsilon \tag{3}
\]

\[
\int_I g - \epsilon \leq \int_I g \leq \int_I g = \int_I g \leq \int_I g \leq \int_I g + \epsilon \tag{4}
\]

From (1) and (2) we get that

\[ f(x) + g(x) \leq f(x) + g(x) \leq \bar{f}(x) + \bar{g}(x) \quad \forall x \in I \]

which implies that we have:

\[
\int_I (f + g) \leq \int_I (f + g) \leq \int_I (f + g) \leq \int_I (f + g) \leq \int_I (f + g) \tag{5}
\]

From (3),(4),(5), and Lemma 1, we get that:

\[
0 \leq \int_I (f + g) - \int_I (f + g) \leq \int_I (\bar{f} + \bar{g}) - \int_I (f + g) = \left( \int_I \bar{f} + \int_I \bar{g} \right) - \left( \int_I f + \int_I g \right)
\]

\[
\leq \left( \int_I f + \epsilon + \int_I g + \epsilon \right) - \left( \int_I f - \epsilon + \int_I g - \epsilon \right) = 4\epsilon
\]

Hence, \(0 \leq \int_I (f + g) - \int_I (f + g) \leq 4\epsilon\). Let \(\epsilon \to 0\) then we get that \(\int_I (f + g) = \int_I (f + g)\) and so \(f + g\) is Riemann integrable.

To show that \(\int_I (f + g) = \int_I f + \int_I g\) let us do the following:
Add (3) and (4), use (5), Lemma 1, and that \(f+g\) is Riemann integrable to get the following inequalities:

\[
\int_I f + \int_I g - 2\epsilon \leq \int_I f + \int_I g
\]

\[
\leq \int_I f + \int_I g = \int_I f + \int_I g = \int_I f + \int_I g
\]

\[
\leq \int_I \bar{f} + \int_I \bar{g} = \int_I f + \int_I g + 2\epsilon
\]
Therefore we have that:
\[
\int_I f + \int_I g - 2\epsilon \leq \int_I (f + g) \leq \int_I f + \int_I g + 2\epsilon
\]
So now we can let \( \epsilon \to 0 \) (since we no longer have anything that depends on \( \epsilon \)) and conclude that \( \int_I (f + g) = \int_I f + \int_I g \).

Thus \( f + g \) is Riemann integrable and \( \int_I (f + g) = \int_I f + \int_I g \)

Now let us prove that for \( c \in \mathbb{R}, c \geq 0 \) and \( f \) Riemann integrable, that \( cf \) is Riemann integrable, and that \( \int_I cf = c \int_I f \).

**Proof.** Since \( f \) is Riemann integrable, we know that given \( \epsilon > 0 \) there are functions, say \( \underline{f}, \overline{f} \) such that

\[
\underline{f}(x) \leq f(x) \leq \overline{f}(x) \quad \forall x \in I
\]

and

\[
\int_I f - \epsilon \leq \int_I \underline{f} \leq \int_I f = \int_I \overline{f} \leq \int_I \overline{f} + \epsilon
\]

Multiplying (6),(7) through by our constant \( c \) we get the following:

\[
c\underline{f}(x) \leq cf(x) \leq c\overline{f}(x) \quad \forall x \in I
\]

\[
c\int_I f - c\epsilon \leq c\int_I \underline{f} \leq c\int_I f = c\int_I \overline{f} \leq c\int_I \overline{f} + c\epsilon
\]

From (8), and **Lemma 1** we know that \( c\underline{f} \) and \( c\overline{f} \) are piecewise constant and so we have:

\[
\int_I c\underline{f} \leq \int_I cf \leq \int_I c\overline{f}
\]

Now from (9),(10), and **Lemma 1**, we get:

\[
0 \leq \int_I cf - \int_I c\underline{f} \leq \int_I c\overline{f} - \int_I c\underline{f} = c\int_I \overline{f} - c\int_I \underline{f} \leq \left( c\int_I f + c\epsilon \right) - \left( c\int_I f - c\epsilon \right) = 2c\epsilon
\]

Hence, \( 0 \leq \int_I cf - \int_I c\underline{f} \leq 2c\epsilon \), let \( \epsilon \to 0 \) and we get that \( \int_I cf = \int_I c\overline{f} \) so we have that \( cf \) is Riemann integrable for \( c \geq 0 \).
Now let us show that $\int_I cf = c \int_I f$

From (9), (10), Lemma 1, and that $cf$ is Riemann integrable we have the following inequalities:

$$c \int_I f - c\epsilon \leq c \int_I f = \int_I cf \leq \int_I cf = \int_I cf \leq \int_I cf = c \int_I f + c\epsilon$$

Therefore, we have:

$$c \int_I f - c\epsilon \leq \int_I cf \leq c \int_I f + c\epsilon$$

Now we can let $\epsilon \to 0$ and conclude that $\int_I cf = c \int_I f$.

Thus, $cf$ is Riemann integrable and $\int_I cf = c \int_I f$.

Now let us prove that $cf$ is Riemann integrable and that $\int_I cf = c \int_I f$ for $c < 0$.

**Proof.** Since $f$ is Riemann integrable, we know that given $\epsilon > 0$ there are functions, say $\underline{f}, \overline{f}$ such that

$$\underline{f}(x) \leq f(x) \leq \overline{f}(x) \forall x \in I \quad (11)$$

and

$$\int_I f - \epsilon \leq \int_I f \leq \int_I f = \int_I f = \int_I \overline{f} \leq \int_I f + \epsilon \quad (12)$$

Multiplying (11), (12) through by our constant $c$ we get the following:

$$c \underline{f}(x) \leq cf(x) \leq c \overline{f}(x) \forall x \in I \quad (13)$$

and

$$c \int_I f + c\epsilon \leq c \int_I f \leq \int_I \overline{f} = c \int_I f = c \int_I f \leq c \int_I f \leq c \int_I f - c\epsilon \quad (14)$$

Note think about flipping the upper and lower integrals.

From (13), and Lemma 1 we know that $\underline{f}$ and $\overline{f}$ are piecewise constant and so we have:

$$\int_I cf \leq \int_I cf \leq \int_I cf \leq \int_I cf \quad (15)$$

Now from (14), (15), and Lemma 1, we get:

$$0 \leq \int_I cf - \int_I cf \leq \int_I cf - \int_I cf = c \int_I f - c \int_I f \leq \left( c \int_I f - c\epsilon \right) - \left( c \int_I f + c\epsilon \right) = -2c\epsilon$$

Hence we have, $0 \leq \int_I cf - \int_I cf \leq -2c\epsilon$, let $\epsilon \to 0$ and we get that $\int_I cf = \int_I cf$ so we have that $cf$ is Riemann integrable for $c < 0$.  


Now let us show that $\int_I cf = c \int_I f$ for $c < 0$.

From (14), (15), Lemma 1, and that $cf$ is Riemann integrable we get the following inequalities:

$$c \int_I f + c \epsilon \leq \int_I cf \leq \int_I cf = \int_I cf \leq \int_I cf = \int_I cf \leq c \int_I f - c \epsilon$$

Therefore we have that:

$$c \int_I f + c \epsilon \leq \int_I cf \leq c \int_I f - c \epsilon$$

Now we have nothing that depends on $\epsilon$, so we can let $\epsilon \to 0$ and conclude that $\int_I cf = c \int_I f$ for $c < 0$. \qed

Together, this proves that $af + bg$ is Riemann integrable, and that $\int_I (af + bg) = a \int_I f + b \int_I g$.