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Math 402
401 Final Exam

Exercise 1: (a) Let $g : X \to \mathbb{R}$, assume f is a bounded function on $X \subset \mathbb{R}$, let x_0 be an adherent point of X. Show that if $\lim_{x\to x_0} g(x) = 0$, then $\lim_{x\to x_0} g(x)f(x) = 0$ as well.

Proof. We know given $\epsilon' > 0$ $\exists \delta' > 0$ such that $|g(x)| \leq \epsilon'$ for $|x - x_0| < \delta$ and also that $\exists M \in \mathbb{R}$ with $|f(x)| \leq M$ $\forall x \in X$ as f is bounded.
Given $\epsilon > 0$ take $\epsilon' = \dfrac{\epsilon}{M} > 0$ then $\exists \delta > 0$ with $|x - x_0| < \delta$ such that $|g(x)| \leq \dfrac{\epsilon}{M}$

$$\begin{aligned}
&\Rightarrow |g(x)|M \leq \epsilon \\
&\Rightarrow |g(x)||f(x)| \leq \epsilon \quad (\text{since } |f(x)| \leq M \ \forall x \in X) \\
&\Rightarrow |g(x)f(x) - 0| \leq \epsilon \\
&\Rightarrow \lim_{x\to x_0} g(x)f(x) = 0
\end{aligned}$$

□

(b) Use the ϵ–δ definition of continuity to show that the linear function $f(x) = ax + b$ is continuous at every point $x \in \mathbb{R}$.

Proof. Given $\epsilon > 0$ take $\delta = \dfrac{\epsilon}{|a|}$ and consider $x \in \mathbb{R}$ such that $|x - x_0| < \delta$

$$\begin{aligned}
&\Rightarrow |x - x_0| < \frac{\epsilon}{|a|} \\
&\Rightarrow |a||x - x_0| < \epsilon \\
&\Rightarrow |ax - ax_0| < \epsilon \\
&\Rightarrow |ax + b - ax_0 - b| < \epsilon \\
&\Rightarrow |(ax + b) - (ax_0 + b)| < \epsilon \\
&\Rightarrow |f(x) - f(x_0)| < \epsilon
\end{aligned}$$

Hence f is continuous at all $x_0 \in \mathbb{R}$. □

Exercise 2: Let $f : \mathbb{R} \to \mathbb{R}$ that satisfies the multiplicative property $f(x + y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$. Assume f is not identically equal to zero.

(i) Show that $f(0) = 1$ and that $f(-x) = \frac{1}{f(x)}$ for all $x \in \mathbb{R}$. Show that $f(x) > 0$ for all $x \in \mathbb{R}$.

First let us prove two simple Lemmas.

Lemma 1. $f(0) \neq 0$

Proof. Suppose $f(0) = 0$ then we have that $f(x) = f(x + 0) = f(x)f(0) = 0$ $\forall x \in \mathbb{R}$ which is a contradiction since f is not identically equal to zero. Hence $f(0) \neq 0$. □

Lemma 2. $f(x) \neq 0$ $\forall x \in \mathbb{R}$.

Proof. Suppose not, that is, assume that there is some x with $f(x) = 0$. Then we have that $f(0) = f(x - x) = f(x)f(-x) \Rightarrow f(0) = 0$ a contradiction from **Lemma 1**. Thus we have that $f(x) \neq 0 \ \forall x \in \mathbb{R}$. $\square$

Now let us prove the results in the exercise

Proof. If we take $x = y = 0$ then we get that $f(0) = f(0)f(0)$ and since $f(0) \neq 0$ we can divide through to get $f(0) = 1$.

For the next part, let us take $y = -x$. Then we have that:
$f(0) = f(x - x) = f(x)f(-x) \Rightarrow 1 = f(x)f(-x) \Rightarrow f(x) = \dfrac{1}{f(-x)}$.

To show that $f(x) > 0 \ \forall x \in \mathbb{R}$ note that $f(x) = f(x/2 + x/2) = f(x/2)f(x/2) = (f(x/2))^2$ and from **Lemma 2** we have that $f(x) \neq 0 \ \forall x \in \mathbb{R}$.
Therefore $f(x) > 0 \ \forall x \in \mathbb{R}$. $\square$

(ii) Let $a = f(1)$ (by (i) $a > 0$). Show that $f(n) = a^n$ for all $n \in \mathbb{N}$. Use (i) to show that $f(z) = a^z$ for all $z \in \mathbb{Z}$.

Proof. (By induction on n)
Base Case: $(n = 0)$.
$a^0 = 1$ and from (i) $f(0) = 1$ this proves the base case.
Inductive Step: Assume for some $n \in \mathbb{N}$ that $f(n) = a^n$, then show true for $n + 1$.
$f(n + 1) = f(n)f(1) = (a^n)a = a^{n+1}$
Thus, by induction, we have that $f(n) = a^n \ \forall n \in \mathbb{N}$.

For the second part note that if $z \geq 0$ then we are done, so assume $z < 0$. Then by definition we have $-z = n$ for some $n \in \mathbb{N}$.
Hence we have, $f(z) = \dfrac{1}{f(-z)} = \dfrac{1}{f(n)} = \dfrac{1}{a^n} = \dfrac{1}{a^{-z}} = a^z$
Thus, we have that $f(z) = a^z \ \forall z \in \mathbb{Z}$. $\square$

(iii) Show that $f(r) = a^r$ for all $r \in \mathbb{Q}$.

Proof. Let $r = p/q \ \ p, q \in \mathbb{Z}, q \neq 0$.
So, $f(1) = f(\underbrace{1/q + 1/q + ... + 1/q}_{\text{q times}}) = f(1/q)f(1/q)...f(1/q) = (f(1/q))^q$
$\Rightarrow (f(1/q))^q = a \Rightarrow f(1/q) = a^{1/q}$
Now $f(p/q) = f(\underbrace{1/q + 1/q + ... + 1/q}_{\text{p times}}) = f(1/q)f(1/q)...f(1/q) = (f(1/q))^p$
$\Rightarrow f(p/q) = (f(1/q))^p = (a^{1/q})^p = a^{p/q} = a^r$
Thus $f(r) = a^r$. $\square$

(iv) Show that if f is continuous at $x = 0$, then f is continuous at every point in $\mathbb{R}$

Proof. Since f is continuous at $x = 0$ we have that $\lim_{x\to 0} f(x) = f(0) = 1$.
From the multiplicative property of f and part (i) we get:

$$\begin{aligned}
\lim_{x\to x_0} \frac{f(x)}{f(x_0)} &= \lim_{x\to x_0} f(x)f(-x_0) \\
&= \lim_{x\to x_0} f(x - x_0) \\
&= \lim_{(x-x_0)\to 0} f(x - x_0) \\
&= f(0) \\
&= 1
\end{aligned}$$

So we have that $\lim_{x\to x_0} \dfrac{f(x)}{f(x_0)} = 1$ which implies that $\lim_{x\to x_0} f(x) = f(x_0)$.
Therefore f is continuous at every point in $\mathbb{R}$. □

(v) Assume f is continuous at zero, use (iii) and (iv) to conclude that $f(x) = a^x$ for all $x \in \mathbb{R}$

Proof. Let $x \in \mathbb{R}$ and let us take a sequence $(r_n)_{n=0}^{\infty}$ of rationals that approaches x. That is, $\lim_{n\to\infty} r_n = x$. From (iv) we have continuity and so we have $\lim_{n\to\infty} f(r_n) = f(x)$. But from (iii) we have that $f(r_n) = a^{r_n}$ hence we get that $\lim_{n\to\infty} f(r_n) = a^x$. Thus $f(x) = a^x$. □

Exercise 3: A function $f : \mathbb{R} \to \mathbb{R}$ satisfies a Lipschitz condition with constant $M > 0$ if for all $x, y \in \mathbb{R}$,

$$|f(x) - f(y)| \leq M|x - y|$$

Assume $h, g : \mathbb{R} \to \mathbb{R}$ each satisfy a Lipschitz condition with constant M_1 and M_2 respectively.

(a) Show that $(h + g)$ satisfies a Lipschitz condition with constant $(M_1 + M_2)$.

Proof.

$$\begin{aligned}
|(h+g)(x) - (h+g)(y)| &= |h(x) + g(x) - (h(y) + g(y))| \\
&= |h(x) - h(y) + g(x) - g(y)| \\
&\leq |h(x) - h(y)| + |g(x) - g(y)| \\
&\leq M_1|x - y| + M_2|x - y| \\
&= (M_1 + M_2)|x - y|
\end{aligned}$$

□

(b) Show that the composition $(h \circ g)$ satisfy a Lipschitz condition. With what constant?

Proof. $|h(g(x)) - h(g(y))| \leq M_1|g(x) - g(y)| \leq M_1M_2|x - y|$.
Constant is M_1M_2 □

(c) Show that the product (hg) does not necessarily satisfy a Lipschitz condition. However, if both functions are bounded then the product satisfies a Lipschitz condition.

Proof. From our Review we have that Lipschitz $\Rightarrow$ Uniformly Continuous. Hence Not Uniformly Continuous $\Rightarrow$ Not Lipschitz.
Using this take $h, g = x$ certainly h, g satisfy Lipschitz, but hg is not uniformly continuous, and hence does not satisfy Lipschitz.

Let h, g be bounded by A, B respectively. That is, $|h(x)| \leq A \; \forall x \in \mathbb{R}$ and $|g(x)| \leq B \; \forall x \in \mathbb{R}$.

$$\begin{aligned}
|h(x)g(x) - h(y)g(y)| &= |h(x)g(x) - h(x)g(y) + h(x)g(y) - h(y)g(y)| \\
&\leq |h(x)||(g(x) - g(y))| + |g(y)||(h(x) - h(y))| \\
&\leq |h(x)|M_2|x - y| + |g(y)|M_1|x - y| \\
&= (M_1|g(y)| + M_2|h(x)|)|x - y| \\
&\leq (M_1B + M_2A)|x - y|
\end{aligned}$$

Hence we have that if h, g are bounded then the product hg satsfies a Lipschitz condition. $\square$

Exercise 4: (a) Assume that $f : [0, \infty) \to \mathbb{R}$ is continuous at every point on its domain. Show that if there exists $b > 0$ such that f is uniformly continuous on the set $[b, \infty)$, then f is uniformly continuous on $[0, \infty)$.

Proof. Since f is continuous on $[0, b]$ we can use **Theorem 9.9.16** to conclude that f is uniformly continuous on $[0, b]$.

And now since f is uniformly continuous on $[0, b]$ we know for every $\epsilon' > 0$ there is some δ' such that $|f(x) - f(y)| < \epsilon'$ whenever $x, y \in [0, b]$ such that $|x - y| < \delta'$

And since f is also uniformly continuous on $[b, \infty)$ we know that for every $\epsilon'' > 0$ there is some δ'' such that $|f(x) - f(y)| < \epsilon''$ whenever $x, y \in [b, \infty)$ such that $|x - y| < \delta''$

Now given $\epsilon > 0$ take $\epsilon' = \epsilon/2$ and $\epsilon'' = \epsilon/2$ and then take $\delta = \min(\delta', \delta'')$ so that $|x - y| < \delta$ for $x, y \in [0, \infty)$.

Now we have three cases:

(i): $x, y \in [0, b]$
(ii): $x, y \in [b, \infty)$
(iii): $x < b < y$

For case (i): Since f is uniformly continuous on $[0, b]$ we have $|f(x) - f(y)| < \epsilon' < \epsilon$

For case (ii): Since f is uniformly continuous on $[b, \infty)$ we have $|f(x) - f(y)| < \epsilon'' < \epsilon$

For case (iii): Since $x, b \in [0, b]$ we have $|f(x) - f(b)| < \epsilon' = \epsilon/2$ and since $b, y \in [b, \infty)$ we have $|f(b) - f(y)| < \epsilon'' = \epsilon/2$

Thus we have:

$$\begin{aligned}
|f(x) - f(y)| &= |f(x) - f(b) + f(b) - f(y)| \\
&\leq |f(x) - f(b)| + |f(b) - f(y)| \quad \text{(triangle inequality)} \\
&< \epsilon/2 + \epsilon/2 \\
&= \epsilon
\end{aligned}$$

So $|f(x) - f(y)| < \epsilon$

In each case we have $|f(x) - f(y)| < \epsilon$ thus f is uniformly continuous on $[0, \infty)$ $\square$

(b) Prove that $f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$.

Proof. First let us show that f is uniformly continuous on $[1, \infty)$.
Given $\epsilon > 0$ take $\delta = \epsilon$ so that $|x - x_0| < \delta = \epsilon$, for $x, x_0 \in [1, \infty)$.

$$\begin{aligned}
&\Rightarrow \frac{|x - x_0|}{\sqrt{x} + \sqrt{x_0}} \leq \frac{\epsilon}{\sqrt{x} + \sqrt{x_0}} < \epsilon \quad (\text{as } \sqrt{x} + \sqrt{x_0} > 1) \\
&\Rightarrow \left| \frac{x - x_0}{\sqrt{x} + \sqrt{x_0}} \right| < \epsilon \\
&\Rightarrow \left| \frac{\sqrt{x} - \sqrt{x_0}}{1} \frac{\sqrt{x} + \sqrt{x_0}}{\sqrt{x} + \sqrt{x_0}} \right| < \epsilon \\
&\Rightarrow |\sqrt{x} - \sqrt{x_0}| < \epsilon
\end{aligned}$$

Thus, $f(x) = \sqrt{x}$ is uniformly continuous on $[1, \infty)$. Now from **Proposition 9.4.11** we know that f is continuous on $[0, \infty)$ so with this and our proof that f is uniformly continuous on $[1, \infty)$ we can use part (a) to conclude that f is uniformly continuous on $[0, \infty)$. $\square$

Exercise 5: Let $\{a_j\}_{j \geq 0}$ be a sequence of real numbers. Assume known that the derivative of $f(x) = e^x$ equals f, that is, f is differentiable on $\mathbb{R}$ and $f'(x) = e^x$.

(a) Show that $f : \mathbb{R} \to (0, \infty)$ is invertible, and that its inverse $f^{-1} : (0, \infty) \to \mathbb{R}$ is differentiable. Find the derivative of the inverse function.

Proof. Since $e > 1$ we know that $f(x) = e^x$ is strictly increasing on $(0, \infty)$ thus we can use **Proposition 9.8.3** to conclude that its inverse exists and is continuous. And since we have that $f'(x) = f(x) \neq 0 \ \forall x \in \mathbb{R}$ we can use the **Inverse Function Theorem** to conclude that f^{-1} has a derivative and that for $f(x) = y$ we have:

$$(f^{-1})'(y) = \frac{1}{f'(x)} = \frac{1}{e^x} = \frac{1}{y}$$

$\square$

(b) Define a function $F_n : \mathbb{R} \to \mathbb{R}$ for each $n \in \mathbb{N}$ by

$$F_n(x) = \begin{cases} \displaystyle\sum_{j=0}^{n} (n - j) x^j e^{-jx} & \text{if } x > 0 \\ \alpha_n x + \beta_n & \text{if } x \leq 0 \end{cases}$$

Can you choose $\alpha_n, \beta_n \in \mathbb{R}$ so that F_n is differentiable on $\mathbb{R}$? Justify your answer.

Proof. First let us note that F_n is clearly continuous for $x > 0$ as we have a finite sum of continuous functions (exponentials) products, and for $x < 0$ since we have a line. The problem is at $x = 0$.
For F_n to be continuous at $x = 0$ we need:

$$\begin{aligned}
&\alpha_n \times 0 + \beta_n = \lim_{x \to 0^+} \sum_{j=0}^{n} (n-j) x^j e^{-jx} \\
&\Rightarrow \beta_n = \lim_{x \to 0^+} \left[(n) x^0 e^0 + \sum_{j=1}^{n} (n-j) x^j e^{-jx} \right] \\
&\Rightarrow \beta_n = n + 0 = n.
\end{aligned}$$

Now we have that F_n is continuous whenever $\beta_n = n$.

For differentiability we will need α_n, the slope of our line (when $x \leq 0$), to have the same slope as our sum when $x \to 0^+$.
That is, we need:

$$\begin{aligned}
&\alpha_n = \frac{d}{dx} \left(\sum_{j=0}^{n} (n-j) x^j e^{-jx} \right)_{x=0} \\
&\Rightarrow \alpha_n = \frac{d}{dx} \left(n + (n-1) x e^{-x} + \sum_{j=2}^{n} (n-j) x^j e^{-jx} \right)_{x=0} \\
&\Rightarrow \alpha_n = \left(0 + (n-1)(e^{-x} - x e^{-x}) + \sum_{j=2}^{n} (n-j)(j x^{j-1} e^{-jx} - j x^j e^{-jx}) \right)_{x=0} \\
&\Rightarrow \alpha_n = n - 1
\end{aligned}$$

So if we take $\alpha_n = n - 1$ and $\beta_n = n$ then we have that F_n is differentiable on $\mathbb{R}$. $\square$

Exercise 6: Let h be a differentiable function defined on the interval $[0, 3]$, and assume that $h(0) = 1, h(1) = 2$ and $h(3) = 2$.

(a) Show that there exists a point $d \in [0, 3]$ such that $h(d) = d$.

Proof. Let us look at $g : [1, 3] \to \mathbb{R}$ defined by $g(x) = h(x) - x$. Note that g is continuous by continuity properties as both h and x are continuous.

$$\begin{aligned}
g(1) &= h(1) - 1 = 2 - 1 = 1 \\
g(3) &= h(3) - 3 = 2 - 3 = -1
\end{aligned}$$

Thus we have that $g(3) \leq 0 \leq g(1)$ so take $y = 0$ and then by the **Intermediate Value Theorem** we know that there exist $d \in [1, 3]$ with $g(d) = y = 0$.
Hence, $h(d) - d = 0 \Rightarrow h(d) = d$ $\square$

(b) Show that there exists a point $c \in (0,3)$ such that $h'(c) = 1/3$.

Proof. This follows directly from the **Mean Value Theorem** which tells us that there is a $c \in (0,3)$ such that $h'(c) = \dfrac{h(3)-h(0)}{3-0} = \dfrac{2-1}{3} = \dfrac{1}{3}$ $\square$

(c) Show that there exists a point $b \in (0,3)$ such that $h'(b) = 1/4$.

Proof. From **Rolle's Thoerem** on the interval $[1,3]$ there is a point $a \in [1,3]$ such that $h'(a) = 0$. Using this and the point c from part (b) we have for $\alpha = 1/4$ that
$h'(a) < \alpha < h'(c)$ hence by **Darboux's Theorem** there is a point $b \in (a,c)$ or $b \in (c,a)$ such that $h'(b) = \alpha = 1/4$. But $(a,c) \subset (0,3)$ or $(c,a) \subset (0,3)$ as $a, c \in (0,3)$.
Therefore $\exists b \in (0,3)$ with $h'(b) = 1/4$. $\square$

Exercise 7: Decide whether the following statements are true or false. Justify your answers with a couple sentences, an example, or a reference.

(a) Continuous functions take bounded closed intervals to bounded closed intervals

True from **Lemma 9.6.3**.

(b) The inverse image of a convergent sequence under a continuous function is necessarily a convergent sequence.

False. Take $f(x) = 1/x$ for $x > 0$, then $f^{-1}(y) = 1/y$. Now if we take $y_n = 1/n \to 0$ (as $n \to \infty$), then $f^{-1}(1/n) = n \to \infty$.

(c) There is a continuous function on an interval that takes exactly two values.

False. Assume there is such a function, say f, and let $f(a) \neq f(b)$ be the two distinct values of f. Since f is continuous we can use the **Intermediate Value Theorem** to get any value y with $f(a) < y < f(b)$ or $f(b) < y < f(a)$ contradicting that the function took on exactly two values.

(d) If f is differentiable on $[a,b]$, then between any two zeroes of f there must be a zero of its derivative f'.

True from **Rolle's Theorem**.

(e) There is a differentiable function at x_0 that is not continuous at x_0.

False by **Proposition 10.1.10**.

Bonus: Let $f : [a,b] \to [a,b]$, assume there is c with $0 < c < 1$ such that

$$|f(x) - f(y)| \leq c|x-y| \ \forall \ x, y \in [a,b]$$

(a) Show that f is uniformly continuous on $[a,b]$.

Proof. Give $\epsilon > 0$ take $\delta = \epsilon/c$
Then for $x, y \in [a,b]$ with $|x-y| < \delta$ we have that $|x-y| < \epsilon/c \Rightarrow c|x-y| < \epsilon$.
But by hypothesis, $|f(x) - f(y)| \leq c|x-y|$ hence $|f(x) - f(y)| < \epsilon$.
Therefore f is uniformly continuous. $\square$

(b) Pick some $y_0 \in [a, b]$ and given y_n define inductively $y_{n+1} = f(y_n)$. Show that the sequence $(y_n)_{n=0}^{\infty}$ is a Cauchy sequence. Show that there is some $y \in [a, b]$ such that $\lim_{n\to\infty} y_n = y$.

Proof. Let $n, m > N$

$$\begin{aligned}
|f(y_{n-1}) - f(y_{m-1})| &= |y_n - y_m| \\
&= |y_n - y_{n-1} + y_{n-1} - y_{n-2} + ... + y_{m+1} - y_m| \\
&\leq c^{n-1}|y_1 - y_0| + c^{n-2}|y_1 - y_0| + ... + c^{n-m}|y_1 - y_0| \\
&= |y_1 - y_0|(c^{n-m} + c^{n-m+1} + ... + c^{n-1})
\end{aligned}$$

But the latter part is the tail end of a geometric series with $|c| < 1$ and thus goes to 0. So with N large enough we can say that $|y_n - y_m| < \epsilon$. Hence $(y_n)_{n=0}^{\infty}$ is Cauchy.

For the second part, we use **Theorem 6.4.18** to get that $(y_n)_{n=0}^{\infty}$ is a convergent sequence and then we can use **Corollary 9.1.17** to conclude that $(y_n)_{n=0}^{\infty}$ converges in $[a, b]$ hence $\exists y \in [a, b]$ with $\lim_{n\to\infty} y_n = y$. □

(c) Prove that y is a fixed point, that is, $f(y) = y$.

Proof. This is a result of part (b).
Since f is continuous we have $\lim_{n\to\infty} f(y_n) = f(y)$
but $(f(y_n)) \sim (y_n)$ hence $\lim_{n\to\infty} f(y_n) = \lim_{n\to\infty}(y_n) = y$
Thus $f(y) = y$. □

(d) Finally, prove that given any $x \in [a, b]$, then the sequence defined inductively by:
$x_0 = x,\ x_{n+1} = f(x_n)$ converges to y as defined in part (b).

Proof. Since y_0 was arbitrarily chosen from $[a, b]$ we know that every sequence defined this way is Cauchy and converges to a fixed point. □