Exercise 1: (a) Let \( g : X \to \mathbb{R} \), assume \( f \) is a bounded function on \( X \subset \mathbb{R} \), let \( x_0 \) be an adherent point of \( X \). Show that if \( \lim_{x \to x_0} g(x) = 0 \), then \( \lim_{x \to x_0} g(x)f(x) = 0 \) as well.

Proof. We know given \( \epsilon' > 0 \) \( \exists \delta' > 0 \) such that \( |g(x)| \leq \epsilon' \) for \( |x - x_0| < \delta \) and also that \( \exists M \in \mathbb{R} \) with \( |f(x)| \leq M \ \forall x \in X \) as \( f \) is bounded.

Given \( \epsilon > 0 \) take \( \epsilon' = \frac{\epsilon}{M} > 0 \) then \( \exists \delta > 0 \) with \( |x - x_0| < \delta \) such that

\[
|g(x)f(x) - 0| < \epsilon
\]

\[\lim_{x \to x_0} g(x)f(x) = 0\]

(b) Use the \( \epsilon-\delta \) definition of continuity to show that the linear function \( f(x) = ax + b \) is continuous at every point \( x \in \mathbb{R} \).

Proof. Given \( \epsilon > 0 \) take \( \delta = \frac{\epsilon}{|a|} \) and consider \( x \in \mathbb{R} \) such that \( |x - x_0| < \delta \)

\[
|a||x - x_0| < \epsilon
\]

\[
|ax - ax_0| < \epsilon
\]

\[
|ax + b - ax_0 - b| < \epsilon
\]

\[
|(ax + b) - (ax_0 + b)| < \epsilon
\]

\[
|f(x) - f(x_0)| < \epsilon
\]

Hence \( f \) is continuous at all \( x_0 \in \mathbb{R} \).

Exercise 2: Let \( f : \mathbb{R} \to \mathbb{R} \) that satisfies the multiplicative property \( f(x + y) = f(x)f(y) \) for all \( x, y \in \mathbb{R} \). Assume \( f \) is not identically equal to zero.

(i) Show that \( f(0) = 1 \) and that \( f(-x) = \frac{1}{f(x)} \) for all \( x \in \mathbb{R} \). Show that \( f(x) > 0 \) for all \( x \in \mathbb{R} \).

First let us prove two simple Lemmas.

Lemma 1. \( f(0) \neq 0 \)

Proof. Suppose \( f(0) = 0 \) then we have that \( f(x) = f(x + 0) = f(x)f(0) = 0 \ \forall x \in \mathbb{R} \) which is a contradiction since \( f \) is not identically equal to zero. Hence \( f(0) \neq 0 \).

Lemma 2. \( f(x) \neq 0 \ \forall x \in \mathbb{R} \).
Proof. Suppose not, that is, assume that there is some \( x \) with \( f(x) = 0 \). Then we have that
\[ f(0) = f(x-x) = f(x)f(-x) \Rightarrow f(0) = 0 \] a contradiction from Lemma 1. Thus we have that \( f(x) \neq 0 \ \forall x \in \mathbb{R} \).

Now let us prove the results in the exercise

Proof. If we take \( x = y = 0 \) then we get that \( f(0) = f(0)f(0) \) and since \( f(0) \neq 0 \) we can divide through to get \( f(0) = 1 \).

For the next part, let us take \( y = -x \). Then we have that:
\[
f(0) = f(x-x) = f(x)f(-x) \Rightarrow 1 = f(x)f(-x) \Rightarrow f(x) = \frac{1}{f(-x)}.
\]

To show that \( f(x) > 0 \ \forall x \in \mathbb{R} \) note that \( f(x) = f(x/2 + x/2) = f(x/2)f(x/2) = (f(x/2))^2 \) and from Lemma 2 we have that \( f(x) \neq 0 \ \forall x \in \mathbb{R} \).
Therefore \( f(x) > 0 \ \forall x \in \mathbb{R} \).

(ii) Let \( a = f(1) \) (by (i) \( a > 0 \)). Show that \( f(n) = a^n \) for all \( n \in \mathbb{N} \). Use (i) to show that \( f(z) = a^z \) for all \( z \in \mathbb{Z} \).

Proof. (By induction on \( n \))
Base Case: \( (n = 0) \).
\( a^0 = 1 \) and from (i) \( f(0) = 1 \) this proves the base case.
Inductive Step: Assume for some \( n \in \mathbb{N} \) that \( f(n) = a^n \), then show true for \( n+1 \).
\[
f(n+1) = f(n)f(1) = (a^n)a = a^{n+1}
\]
Thus, by induction, we have that \( f(n) = a^n \ \forall n \in \mathbb{N} \).

For the second part note that if \( z \geq 0 \) then we are done, so assume \( z < 0 \). Then by definition we have \( -z = n \) for some \( n \in \mathbb{N} \).
Hence we have, \( f(z) = \frac{1}{f(-z)} = \frac{1}{f(n)} = \frac{1}{a^n} = \frac{1}{a^{-z}} = a^z \)
Thus, we have that \( f(z) = a^z \ \forall z \in \mathbb{Z} \).

(iii) Show that \( f(r) = a^r \) for all \( r \in \mathbb{Q} \).

Proof. Let \( r = p/q \ p, q \in \mathbb{Z}, q \neq 0 \)
So, \( f(1) = f(\underbrace{1/q + 1/q + ... + 1/q}_{q \ \text{times}}) = f(1/q)f(1/q)...f(1/q) = (f(1/q))^q \)
\[
\Rightarrow (f(1/q))^q = a \Rightarrow f(1/q) = a^{1/q}
\]
Now \( f(p/q) = f(\underbrace{1/q + 1/q + ... + 1/q}_{p \ \text{times}}) = f(1/q)f(1/q)...f(1/q) = (f(1/q))^p \)
\[
\Rightarrow f(p/q) = (f(1/q))^p = (a^{1/q})^p = a^{p/q} = a^r
\]
Thus \( f(r) = a^r \).

(iv) Show that if \( f \) is continuous at \( x = 0 \), then \( f \) is continuous at every point in \( \mathbb{R} \)
Proof. Since $f$ is continuous at $x = 0$ we have that $\lim_{x \to 0} f(x) = f(0) = 1$.
From the multiplicative property of $f$ and part (i) we get:

$$
\lim_{x \to x_0} \frac{f(x)}{f(x_0)} = \lim_{x \to x_0} f(x)f(-x_0) = \lim_{x \to x_0} f(x - x_0) = \lim_{(x-x_0) \to 0} f(x - x_0) = f(0) = 1
$$

So we have that $\lim_{x \to x_0} \frac{f(x)}{f(x_0)} = 1$ which implies that $\lim_{x \to x_0} f(x) = f(x_0)$.

Therefore $f$ is continuous at every point in $\mathbb{R}$. 

(v) Assume $f$ is continuous at zero, use (iii) and (iv) to conclude that $f(x) = ax$ for all $x \in \mathbb{R}$

Proof. Let $x \in \mathbb{R}$ and let us take a sequence $(r_n)_{n=0}^\infty$ of rationals that approaches $x$. That is, $\lim_{n \to \infty} r_n = x$. From (iv) we have continuity and so we have $\lim_{n \to \infty} f(r_n) = f(x)$. But from (iii) we have that $f(r_n) = a^{r_n}$ hence we get that $\lim_{n \to \infty} f(r_n) = a^x$. Thus $f(x) = a^x$. 

Exercise 3: A function $f : \mathbb{R} \to \mathbb{R}$ satisfies a Lipschitz condition with constant $M > 0$ if for all $x, y \in \mathbb{R}$,

$$
|f(x) - f(y)| \leq M|x - y|
$$

Assume $h, g : \mathbb{R} \to \mathbb{R}$ each satisfy a Lipschitz condition with constant $M_1$ and $M_2$ respectively.

(a) Show that $(h + g)$ satisfies a Lipschitz condition with constant $(M_1 + M_2)$.

Proof.

$$\begin{align*}
|(h + g)(x) - (h + g)(y)| &= |h(x) + g(x) - (h(y) + g(y))| \\
&= |h(x) - h(y) + g(x) - g(y)| \\
&\leq |h(x) - h(y)| + |g(x) - g(y)| \\
&\leq M_1|x - y| + M_2|x - y| \\
&= (M_1 + M_2)|x - y|
\end{align*}$$

(b) Show that the composition $(h \circ g)$ satisfy a Lipschitz condition. With what constant?

Proof. $|h(g(x)) - h(g(y))| \leq M_1|g(x) - g(y)| \leq M_1M_2|x - y|$.

Constant is $M_1M_2$

(c) Show that the product $(hg)$ does not necessarily satisfy a Lipschitz condition. However, if both functions are bounded then the product satisfies a Lipschitz condition.
Proof. From our Review we have that Lipschitz \( \Rightarrow \) Uniformly Continuous. Hence Not Uniformly Continuous \( \Rightarrow \) Not Lipschitz.
Using this take \( h, g = x \) certainly \( h, g \) satisfy Lipschitz, but \( hg \) is not uniformly continuous, and hence does not satisfy Lipschitz.

Let \( h, g \) be bounded by \( A, B \) respectively. That is, \( |h(x)| \leq A \forall x \in \mathbb{R} \) and \( |g(x)| \leq B \forall x \in \mathbb{R} \).

\[
|h(x)g(x) - h(y)g(y)| = |h(x)g(x) - h(x)g(y) + h(x)g(y) - h(y)g(y)|
\leq |h(x)||g(x) - g(y)|| + |g(y)||h(x) - h(y)||
\leq |h(x)||M_2|x - y| + |g(y)||M_1|x - y|
= (M_1|g(y)| + M_2|h(x)||)|x - y|
\leq (M_1B + M_2A)|x - y|
\]

Hence we have that if \( h, g \) are bounded then the product \( hg \) satisfies a Lipschitz condition. \( \square \)

Exercise 4: (a) Assume that \( f : [0, \infty) \rightarrow \mathbb{R} \) is continuous at every point on its domain. Show that if there exists \( b > 0 \) such that \( f \) is uniformly continuous on the set \( [b, \infty) \), then \( f \) is uniformly continuous on \( [0, \infty) \).

Proof. Since \( f \) is continuous on \( [0, b] \) we can use Theorem 9.9.16 to conclude that \( f \) is uniformly continuous on \( [0, b] \).

And now since \( f \) is uniformly continuous on \( [0, b] \) we know for every \( \varepsilon' > 0 \) there is some \( \delta' \) such that \( |f(x) - f(y)| < \varepsilon' \) whenever \( x, y \in [0, b] \) such that \( |x - y| < \delta' \)

And since \( f \) is also uniformly continuous on \( [b, \infty) \) we know that for every \( \varepsilon'' > 0 \) there is some \( \delta'' \) such that \( |f(x) - f(y)| < \varepsilon'' \) whenever \( x, y \in [b, \infty) \) such that \( |x - y| < \delta'' \)

Now given \( \varepsilon > 0 \) take \( \varepsilon' = \varepsilon/2 \) and \( \varepsilon'' = \varepsilon/2 \) and then take \( \delta = \min(\delta', \delta'') \) so that \( |x - y| < \delta \) for \( x, y \in [0, \infty) \).

Now we have three cases:
(i): \( x, y \in [0, b] \)
(ii): \( x, y \in [b, \infty) \)
(iii): \( x < b < y \)

For case (i): Since \( f \) is uniformly continuous on \( [0, b] \) we have \( |f(x) - f(y)| < \varepsilon' < \varepsilon \)

For case (ii): Since \( f \) is uniformly continuous on \( [b, \infty) \) we have \( |f(x) - f(y)| < \varepsilon'' < \varepsilon \)

For case (iii): Since \( x, b \in [0, b] \) we have \( |f(x) - f(b)| < \varepsilon' = \varepsilon/2 \) and since \( b, y \in [b, \infty) \) we have \( |f(b) - f(y)| < \varepsilon'' = \varepsilon/2 \)

Thus we have:
\[
|f(x) - f(y)| = |f(x) - f(b) + f(b) - f(y)|
\leq |f(x) - f(b)| + |f(b) - f(y)| \quad (\text{triangle inequality})
\leq \varepsilon/2 + \varepsilon/2
= \varepsilon
\]
So \(|f(x) - f(y)| < \epsilon\)

In each case we have \(|f(x) - f(y)| < \epsilon\) thus \(f\) is uniformly continuous on \([0, \infty)\). \(\square\)

(b) Prove that \(f(x) = \sqrt{x}\) is uniformly continuous on \([0, \infty)\).

**Proof.** First let us show that \(f\) is uniformly continuous on \([1, \infty)\).

Given \(\epsilon > 0\) take \(\delta = \epsilon\) so that \(|x - x_0| < \delta\), for \(x, x_0 \in [1, \infty)\).

\[
\Rightarrow \frac{|x - x_0|}{\sqrt{x} + \sqrt{x_0}} < \epsilon \quad \text{(as } \sqrt{x} + \sqrt{x_0} > 1)\]

\[
\Rightarrow \left| \frac{x - x_0}{\sqrt{x} + \sqrt{x_0}} \right| < \epsilon
\]

\[
\Rightarrow \left| \frac{\sqrt{x} - \sqrt{x_0}}{\sqrt{x} + \sqrt{x_0}} \right| < \epsilon
\]

\[
\Rightarrow |\sqrt{x} - \sqrt{x_0}| < \epsilon
\]

Thus, \(f(x) = \sqrt{x}\) is uniformly continuous on \([1, \infty)\). Now from **Proposition 9.4.11** we know that \(f\) is continuous on \([0, \infty)\) so with this and our proof that \(f\) is uniformly continuous on \([1, \infty)\) we can use part (a) to conclude that \(f\) is uniformly continuous on \([0, \infty)\). \(\square\)

**Exercise 5:** Let \(\{a_j\}_{j \geq 0}\) be a sequence of real numbers. Assume known that the derivative of \(f(x) = e^x\) equals \(f\), that is, \(f\) is differentiable on \(\mathbb{R}\) and \(f'(x) = e^x\).

(a) Show that \(f : \mathbb{R} \rightarrow (0, \infty)\) is invertible, and that its inverse \(f^{-1} : (0, \infty) \rightarrow \mathbb{R}\) is differentiable. Find the derivative of the inverse function.

**Proof.** Since \(e > 1\) we know that \(f(x) = e^x\) is strictly increasing on \((0, \infty)\) thus we can use **Proposition 9.8.3** to conclude that its inverse exists and is continuous. And since we have that \(f'(x) = f(x) \neq 0 \ \forall x \in \mathbb{R}\) we can use the **Inverse Function Theorem** to conclude that \(f^{-1}\) has a derivative and that for \(f(x) = y\) we have:

\[
(f^{-1})'(y) = \frac{1}{f'(x)} = \frac{1}{e^x} = \frac{1}{y}
\]

\(\square\)

(b) Define a function \(F_n : \mathbb{R} \rightarrow \mathbb{R}\) for each \(n \in \mathbb{N}\) by

\[
F_n(x) = \begin{cases} 
\sum_{j=0}^{n} (n-j)x^je^{-jx} & \text{if } x > 0 \\
\alpha_n x + \beta_n & \text{if } x \leq 0
\end{cases}
\]

Can you choose \(\alpha_n, \beta_n \in \mathbb{R}\) so that \(F_n\) is differentiable on \(\mathbb{R}\)? Justify your answer.
Proof. First let us note that $F_n$ is clearly continuous for $x > 0$ as we have a finite sum of continuous functions (exponentials) products, and for $x < 0$ since we have a line. The problem is at $x = 0$.

For $F_n$ to be continuous at $x = 0$ we need:

$$\alpha_n \times 0 + \beta_n = \lim_{x \to 0^+} \sum_{j=0}^{n} (n-j)x^je^{-jx}$$

$$\Rightarrow \beta_n = \lim_{x \to 0^+} \left[ (n)x^0e^0 + \sum_{j=1}^{n} (n-j)x^je^{-jx} \right]$$

$$\Rightarrow \beta_n = n + 0 = n.$$ 

Now we have that $F_n$ is continuous whenever $\beta_n = n$.

For differentiability we will need $\alpha_n$, the slope of our line (when $x \leq 0$), to have the same slope as our sum when $x \to 0^+$.

That is, we need:

$$\alpha_n = \frac{d}{dx} \left( \sum_{j=0}^{n} (n-j)x^je^{-jx} \right)_{x=0}$$

$$\Rightarrow \alpha_n = \frac{d}{dx} \left( n + (n-1)x^{-1}e^{x-1} + \sum_{j=2}^{n} (n-j)x^j e^{-jx} \right)_{x=0}$$

$$\Rightarrow \alpha_n = \left( 0 + (n-1)(e^{-1} - xe^{-x}) + \sum_{j=2}^{n} (n-j)\left( jx^{j-1}e^{-jx} - jx^je^{-jx} \right) \right)_{x=0}$$

$$\Rightarrow \alpha_n = n - 1$$

So if we take $\alpha_n = n - 1$ and $\beta_n = n$ then we have that $F_n$ is differentiable on $\mathbb{R}$. $\Box$

Exercise 6: Let $h$ be a differentiable function defined on the interval $[0, 3]$, and assume that $h(0) = 1$, $h(1) = 2$ and $h(3) = 2$.

(a) Show that there exists a point $d \in [0, 3]$ such that $h(d) = d$.

Proof. Let us look at $g : [1, 3] \to \mathbb{R}$ defined by $g(x) = h(x) - x$. Note that $g$ is continuous by continuity properties as both $h$ and $x$ are continuous.

$$g(1) = h(1) - 1 = 2 - 1 = 1$$

$$g(3) = h(3) - 3 = 2 - 3 = -1$$

Thus we have that $g(3) \leq 0 \leq g(1)$ so take $y = 0$ and then by the Intermediate Value Theorem we know that there exist $d \in [1, 3]$ with $g(d) = y = 0$.

Hence, $h(d) - d = 0 \Rightarrow h(d) = d$ $\Box$
(b) Show that there exists a point \( c \in (0, 3) \) such that \( h'(c) = 1/3 \).

**Proof.** This follows directly from the **Mean Value Theorem** which tells us that there is a \( c \in (0, 3) \) such that

\[
h'(c) = \frac{h(3) - h(0)}{3 - 0} = \frac{2 - 1}{3} = \frac{1}{3}
\]

(c) Show that there exists a point \( b \in (0, 3) \) such that \( h'(b) = 1/4 \).

**Proof.** From **Rolle’s Theorem** on the interval \([1, 3]\) there is a point \( a \in [1, 3] \) such that

\[
h'(a) = 0
\]

Using this and the point \( c \) from part (b) we have for \( \alpha = 1/4 \) that

\[
h'(a) < \alpha < h'(c)
\]

hence by **Darboux’s Theorem** there is a point \( b \in (a, c) \) or \( b \in (c, a) \) such that \( h'(b) = \alpha = 1/4 \). But \( (a, c) \subset (0, 3) \) or \( (c, a) \subset (0, 3) \) as \( a, c \in (0, 3) \).

Therefore \( \exists b \in (0, 3) \) with \( h'(b) = 1/4 \). □

**Exercise 7:** Decide whether the following statements are true or false. Justify your answers with a couple sentences, an example, or a reference.

(a) Continuous functions take bounded closed intervals to bounded closed intervals

**True** from Lemma 9.6.3.

(b) The inverse image of a convergent sequence under a continuous function is necessarily a convergent sequence.

**False.** Take \( f(x) = 1/x \) for \( x > 0 \), then \( f^{-1}(y) = 1/y \). Now if we take \( y_n = 1/n \to 0 \) (as \( n \to \infty \)), then \( f^{-1}(1/n) = n \to \infty \).

(c) There is a continuous function on an interval that takes exactly two values.

**False.** Assume there is such a function, say \( f \), and let \( f(a) \neq f(b) \) be the two distinct values of \( f \). Since \( f \) is continuous we can use the **Intermediate Value Theorem** to get any value \( y \) with \( f(a) < y < f(b) \) or \( f(b) < y < f(a) \) contradicting that the function took on exactly two values.

(d) If \( f \) is differentiable on \([a, b]\), then between any two zeroes of \( f \) there must be a zero of its derivative \( f' \).

**True** from Rolle’s Theorem.

(e) There is a differentiable function at \( x_0 \) that is not continuous at \( x_0 \).

**False** by Proposition 10.1.10.

**Bonus:** Let \( f : [a, b] \to [a, b] \), assume there is \( c \) with \( 0 < c < 1 \) such that

\[
|f(x) - f(y)| \leq c|x - y| \forall x, y \in [a, b]
\]

(a) Show that \( f \) is uniformly continuous on \([a, b]\).

**Proof.** Give \( \epsilon > 0 \) take \( \delta = \epsilon/c \)

Then for \( x, y \in [a, b] \) with \( |x - y| < \delta \) we have that \( |x - y| < \epsilon/c \Rightarrow c|x - y| < \epsilon \).

But by hypothesis, \( |f(x) - f(y)| \leq c|x - y| \) hence \( |f(x) - f(y)| < \epsilon \).

Therefore \( f \) is uniformly continuous. □
(b) Pick some \( y_0 \in [a, b] \) and given \( y_n \) define inductively \( y_{n+1} = f(y_n) \). Show that the sequence \((y_n)_{n=0}^\infty\) is a Cauchy sequence. Show that there is some \( y \in [a, b] \) such that \( \lim_{n \to \infty} y_n = y \).

**Proof.** Let \( n, m > N \)

\[
|f(y_{n-1}) - f(y_{m-1})| = |y_n - y_m| \\
= |y_n - y_{n-1} + y_{n-1} - y_{n-2} + \ldots + y_{m+1} - y_m| \\
\leq c^{n-1}|y_1 - y_0| + c^{n-2}|y_1 - y_0| + \ldots + c^{n-m}|y_1 - y_0| \\
= |y_1 - y_0|(c^{n-m} + c^{n-m+1} + \ldots + c^{n-1})
\]

But the latter part is the tail end of a geometric series with \( |c| < 1 \) and thus goes to 0. So with \( N \) large enough we can say that \( |y_n - y_m| < \epsilon \). Hence \((y_n)_{n=0}^\infty\) is Cauchy.

For the second part, we use **Theorem 6.4.18** to get that \((y_n)_{n=0}^\infty\) is a convergent sequence and then we can use **Corollary 9.1.17** to conclude that \((y_n)_{n=0}^\infty\) converges in \([a, b]\) hence \( \exists y \in [a, b] \) with \( \lim_{n \to \infty} y_n = y \). \[\square\]

(c) Prove that \( y \) is a fixed point, that is, \( f(y) = y \).

**Proof.** This is a result of part (b).

Since \( f \) is continuous we have \( \lim_{n \to \infty} f(y_n) = f(y) \) but \( (f(y_n)) \sim (y_n) \) hence \( \lim_{n \to \infty} f(y_n) = \lim_{n \to \infty} y_n = y \) Thus \( f(y) = y \). \[\square\]

(d) Finally, prove that given any \( x \in [a, b] \), then the sequence defined inductively by:

\[
x_0 = x, \quad x_{n+1} = f(x_n)
\]

converges to \( y \) as defined in part (b).

**Proof.** Since \( y_0 \) was arbitrarily chosen from \([a, b]\) we know that every sequence defined this way is Cauchy and converges to a fixed point. \[\square\]