Theorem 11.4.1 (g)

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We first prove the following lemma

**Lemma 1** Let $I$ be a bounded interval, and let $f : I \to \mathbb{R}$ be a Riemann integrable function on $I$ ($f \in \mathcal{R}(I)$) that is also a step or piecewise constant function (p.c.f.) with respect to a partition $P_I$. Let $J$ be a bounded interval containing $I$ (i.e. $I \subseteq J$), then

$$F(x) := \begin{cases} f(x) & \text{if } x \in I \\ 0 & \text{if } x \notin I \end{cases}$$

is a p.c.f. with respect to a partition $P_J = P_I \cup \{J \setminus I\}$. Moreover $F \in \mathcal{R}(J)$ and $\int_J F = \int_I f$.

*Proof:* If $f(x)$ is a p.c.f. with respect to a partition $P_I$ then $f(x) = c_K$ for all $x \in K \in P_I$, where $c_K$ denotes a constant. Then by assumption $F(x) = f(x) = c_K$ for all $x \in K \in P_I$ and $F(x) = 0$ for all $x \in K \in \{J \setminus I\}$, i.e. $F(x) = c_K$ for all $x \in K \in P_I \cup \{J \setminus I\}$ where $c_K = c_K$ if $K \in P_I$ and $c_K = 0$ if $K \in \{J \setminus I\}$. Since $P_I \cap \{J \setminus I\} = \emptyset$, $F(x)$ is a p.c.f with respect to $P_I \cup \{J \setminus I\} = P_J$.

Consider

$$\int_I f = \sum_{K \in P_I} c_K = \sum_{K \in P_I \cup \{J \setminus I\}} c_K + \sum_{K \in \{J \setminus I\}} 0 = \sum_{K \in P_J} c_K = \int_J F$$

(The first equality follows from the definition of Riemann integral for a p.c.f., the second from adding zero, the third from the definition of $F$ and because $P_I \cap \{J \setminus I\} = \emptyset$ and $P_I \cup \{J \setminus I\} = P_J$, and the forth from the definition of the Riemann integral for a p.c.f.) Thus $F \in \mathcal{R}(J)$ and $\int_J F = \int_I f$.

We now prove the main theorem.

**Theorem 1 (Theorem 11.4.1 (g))** Let $I$ be a bounded interval, and let $f : I \to \mathbb{R}$ be a Riemann integrable function on $I$ ($f \in \mathcal{R}(I)$). Let $J$ be a bounded interval containing $I$ (i.e. $I \subseteq J$), and let $F : J \to \mathbb{R}$ be the function

$$F(x) := \begin{cases} f(x) & \text{if } x \in I \\ 0 & \text{if } x \notin I \end{cases}$$

then $F$ is Riemann integrable on $J$ and $\int_J F = \int_I f$.

*Proof:* Given $\epsilon > 0$, there exists two p.c.f. $\underline{f}$ and $\overline{f}$ such that $\underline{f} \leq f \leq \overline{f}$ (note that $\underline{f}$ and $\overline{f}$ depend on $\epsilon > 0$) and

$$\int_J \underline{f} \geq \int_J f - \epsilon \quad \text{and} \quad \int_J \overline{f} \leq \int_J f + \epsilon \quad (1)$$

Let

$$\underline{F}(x) := \begin{cases} \underline{f}(x) & \text{if } x \in I \\ 0 & \text{if } x \in J \setminus I \end{cases} \quad \text{and} \quad \overline{F}(x) := \begin{cases} \overline{f}(x) & \text{if } x \in I \\ 0 & \text{if } x \in J \setminus I \end{cases}$$

Note that $\underline{F}(x) \leq F(x) \leq \overline{F}(x), \forall x \in J$ and

$$\int_J \underline{F} = \int_J \underline{f} + \int_{J \setminus I} 0 = \int_J \underline{E} \quad (2)$$

where the first inequality follows from adding zero and the second from Lemma 1. Similarly

$$\int_J \overline{F} = \int_J \overline{f} + \int_{J \setminus I} 0 = \int_J \overline{F} \quad (3)$$
Also note that
\[ \int_J F \leq \sup_{\{F \leq \bar{F}, \text{p.c.t.}\}} \left[ \int_J F \right] = \int_J F \quad \text{and} \quad \int_J F \geq \inf_{\{F \leq \bar{F}, \text{p.c.t.}\}} \left[ \int_J F \right] = \int_J \bar{F} \] (4)
and that (by Lemma 11.3.3)
\[ \int_J F \leq \int_J \bar{F} \] (5)

Putting together equations (1), (2), (3), (4), and (5) we obtain
\[ \int_J f - \epsilon \leq \int_J F = \int_J \bar{F} \leq \int_J F \leq \int_J \bar{F} = \int_J \bar{F} \leq \int_I f + \epsilon \] (6)

This implies that
\[ \int_I f - \epsilon \leq \int_J F \leq \int_I f + \epsilon \]

Since this inequality is true for all \( \epsilon \), this implies
\[ \int_I f = \int_J F = \int_J \bar{F}, \]
which means that \( F \in \mathcal{R}(J) \) and that \( \int_I f = \int_J F. \) \( \blacksquare \)