SOLUTIONS: REVIEW FOR FINAL EXAM - MATH 401/501 - Spring 2016

Instructor: C. Pereyra

1. (a) Show that every bounded sequence in $\mathbb{R}$ has at least a convergent subsequence.

   SOLUTION: Let $\{a_n\}_{n\geq 1}$ be a bounded sequence of real numbers. Let $L = \limsup\{a_n\}$ (could have chosen L to be the $\liminf\{a_n\}$). There is $M > 0$ such that for all $n \geq 1$ then $-M \leq a_n \leq M$, necessarily $-M \leq L \leq M$ (why?[1]). In particular L is a finite real number and it is a limit point of the sequence $\{a_n\}_{n\geq 1}$ if and only if L is a subsequential limit of $\{a_n\}_{n\geq 1}$ (Proposition 6.6.6), that is there is subsequence $\{a_{n_k}\}_{k\geq 1}$ such that $L = \lim_{k\to\infty} a_{n_k}$. (This result is known as the Bolzano-Weiertrass Theorem.)

   (b) Show that given non-empty subset A of real numbers the following are equivalent:

   (i) A is bounded and closed,

   (ii) every sequence in A has a convergent subsequence converging to a point in A.

   SOLUTION: (i)⇒(ii) Assume the non-empty set $A \subset \mathbb{R}$ is bounded and closed, so there exists $M > 0$ such that for all $x \in A$ we have $|x| \leq M$ and A contains all its adherent points, that is $A = \overline{A}$. Given a sequence $\{x_n\}_{n\geq 1}$ in A it is necessarily bounded, because $x_n \in A$ for all $n \geq 1$ therefore $|x_n| \leq M$ for all $n \geq 1$. By part (a) there is a convergent subsequence $\{x_{n_k}\}_{k\geq 1}$ to L, that means that L is an adherent point of A because given $\epsilon > 0$ there is $N > 0$ such that for all $k \geq N$ we have $|x_{n_k} - L| \leq \epsilon$ and $x_{n_k} \in A$. Finally $L \in A$ because A is closed and therefore contains all its adherent points.

   (ii)⇒(i) Assume now that every sequence in A has a convergent subsequence converging to a point in A. We will proceed by contradiction, assume (i) doesn't hold, that is either A is not bounded or A is not closed.

   If A is not bounded then given $n > 0$ there is $x_n \in A$ such that $|x_n| > n$ otherwise n will be a bound for A. We now have a sequence $\{x_n\}_{n\geq 1}$ in A that cannot have a convergent subsequence why?[2]) contradicting (ii).

   If A is bounded but not closed, then there is an adherent point x_0 of A that is not an element of A. In that case x_0 must be a limit point of A, that is there is a sequence $\{x_n\}_{n\geq 1}$ in A that converges to x_0, but all subsequences of this convergent sequence must converge to x_0 which is not in A, contradicting (ii). (This result is known as the Heine-Borel Theorem in $\mathbb{R}$. Sets that have property (ii) are called "compact sets", and the Heine-Borel Theorem says: the compact sets in $\mathbb{R}$ are the closed and bounded sets. In other settings compact sets will be closed and bounded but not all closed and bounded sets will be compact.)

2. Show that any function f with domain the integers $\mathbb{Z}$ will necessarily be continuous at every point on its domain. More generally, show that if $f : X \to \mathbb{R}$, and x_0 is an isolated point of $X \subset \mathbb{R}$, then f is continuous at x_0.

   SOLUTION: $f : \mathbb{Z} \to \mathbb{R}$, given and integer $n_0 \in \mathbb{Z}$, given $\epsilon > 0$ let $\delta = 1/2$. If $n \in \mathbb{Z}$ and $|n - n_0| \leq \delta = 1/2$ then $n = n_0$ (why?[3]), therefore

   $$|f(n) - f(n_0)| = |f(n_0) - f(n_0)| = 0 \leq \epsilon$$

   for all $|n - n_0| \leq \delta = 1/2$. Thus f is continuous at n_0 for all $n_0 \in \mathbb{Z}$.

   More generally if x_0 is an isolated point of $X \subset \mathbb{R}$, then there is a $\delta_0 > 0$ such that if $x \in X$ and $|x - x_0| \leq \delta_0$ then $x = x_0$. Therefore if $f : X \to \mathbb{R}$ and given $\epsilon > 0$ let $\delta = \delta_0$, then

   $$|f(x) - f(x_0)| = |f(x_0) - f(x_0)| = 0 \leq \epsilon$$

   for all $x \in X$ and $|x - x_0| \leq \delta_0 = \delta$. Thus f is continuous at x_0 for all x_0 isolated points of X.

[1] By comparison principle Lemma 6.4.13 in the book we have $-M = \limsup(-M) \leq \limsup\{a_n\} = L \leq \limsup M = M$.

[2] Indeed, suppose there is a convergent subsequence $\lim_{k\to\infty} x_{n_k} = L$ to a point $L \in A$ then given $\epsilon = 1$ there is $N > 0$ such that $|x_{n_k} - L| \leq 1$ for all $k \geq N$. Apply the reverse triangle inequality to get $|x_{n_k}| \leq |L| + 1$, but by construction $|x_{n_k}| \geq n_k$ and the sequence $\{n_k\}_{k\geq 1}$ is a strictly increasing sequence of natural numbers increasing towards infinity, so there is $K \geq N$ such that $|L| + 1 < n_K$ reaching the contradiction $x_{n_K} < x_{n_K}$.

[3] If $n \neq n_0$ then $1 \leq |n - n_0| \leq 1/2$ and $1 \leq 1/2$, this is a contradiction.

3. For each choice of subsets A_i of the real numbers: Is the set bounded or not? Does it have a least upper bound or a greatest lower bound? Find them. Is the set closed or not? Find its closure.

(a) $A_1 = [0,1]$, (b) $A_2 = (0,1]$, (c) $A_3 = \{1/n : n \in \mathbb{N}\backslash\{0\}\}$, (d) $A_4 = \mathbb{Z}$.

SOLUTION: Recall that a set A is closed if it equals its closure $\overline{A}$, the set of all adherent points of A. A set A is always a subset of its closure $\overline{A}$ (why?[4]). Recall also that a point x_0 is NOT an adherent point of A if there is a $\delta > 0$ such that for all $x \in A$ we have $|x - x_0| > \delta$.

(a) The set A_1 is bounded above by 1 and below by 0, since $x \in A_1$ iff $0 \le x \le 1$,

The set A_1 has both a least upper bound, $\sup A_1 = 1$, and a greatest lower bound, $\inf A_1 = 0$. Since both the lower bound $x = 0$ and the upper bound $x = 1$ are in A_1, any lower bound has to be less than or equal to 0 and any upper bound has to be larger or equal to 1, therefore $\inf A_1 = 0$ and $\sup A_1 = 1$.

The set A_1 equals to its closure: $\overline{A_1} = A_1$. Indeed, $A_1 \subset \overline{A_1}$ and it remains to be shown that if $x_0 \in \mathbb{R}\backslash A_1$ then x_0 is NOT an adherent point of A_1, that is x_0 is not in $\overline{A_1}$. Now let $\delta = |x_0|/2 > 0$ if $x_0 < 0$ and let $\delta = (1 - x_0)/2 > 0$ if $x_0 > 1$, in both cases if $x \in A_1 = [0,1]$ then $|x - x_0| > \delta$ (why? draw a picture), therefore x_0 is not an adherent point of A_1, and we are done.

(b) The set A_2 is bounded above by 1 and below by 0. Since $x \in A_2$ iff $0 < x \le 1$,

The set A_2 has both a least upper bound, $\sup A_2 = 1$, and a greatest lower bound, $\inf A_2 = 0$. Since the upper bound $x = 1$ is in A_2 any upper bound has to be larger or equal to 1, so $\sup A_2 = 1$. If $m > 0$ let $x = m/2 > 0$ then $x \in A_2$ and $x < m$, so m cannot be a lower bound for A_2, hence $\inf A_2 = 0$.

The set A_2 is not closed. Its closure is $\overline{A_2} = A_2 \cup \{0\} = A_1$. By part (a) A_1 is closed, so all we need to verify is that $x_0 = 0$ is an adherent point for A_2. Given $\delta > 0$ let $x = \min\{\delta/2, 1\} > 0$ then $x \in A_2$ since by definition $0 < x \le 1$, and also by definition $|x - x_0| = |x| \le \delta/2 < \delta$, therefore x_0 is an adherent point for A_2.

(c) The set A_3 is bounded above by 1 and below by 0. Since if $0 < 1/n \le 1$ for all $0 < n \in \mathbb{N}$.

The set A_3 has both a least upper bound, $\sup A_3 = 1$, and a greatest lower bound, $\inf A_3 = 0$. Since $x = 1$ is in A_3 any upper bound has to be larger or equal to 1, so $\sup A_3 = 1$. Given any $0 < m$ by Archimedean Principle there is $n \in \mathbb{N}$ such that $0 < 1/n < m$, since $1/n \in A_3$, then $m > 0$ cannot be a lower bound for A_3, hence $\inf A_3 = 0$.

The set A_3 is not closed. Its closure is $\overline{A_3} = A_3 \cup \{0\}$. We know $A_3 \subset \overline{A_3}$. we need to show that the only point not in A_3 that is adherent is $x_0 = 0$. First, given $\delta > 0$ by Archimedean Principle there is $n \in \mathbb{N}$ such that $0 < 1/n < \delta$, since $x = 1/n \in A_3$ and $|x - x_0| = 1/n \le \delta$ then $x_0 = 0$ is an adherent point of A_3. Second, we must show that any other point $x_0 \in A_3 \cup \{0\}$ is NOT an adherent point. There are three cases: (1) $x_0 < 0$, (2) $x_0 > 1$, (3) $x_0 \neq 1/n$ for all $0 < n \in \mathbb{N}$ and $0 < x_0 < 1$. In the third case there is a unique $0 < N \in \mathbb{N}$ such that $1/(N+1) < x_0 < 1/N$ (why?[5]). In all three cases, there is room separating x_0 from $A_3 \cup \{0\}$. More precisely, in the first two cases, let $\delta := \min\{|x_0|/2, |x_0 - 1|/2\} > 0$, if $|x - x_0| \le \delta$ then if $x_0 < 0$ we have $x < 0$ and if $x_0 > 1$ we have $x > 1$, in either case x is not in A_3, this takes care of cases (1) and (2). In case (3), let $\delta := \min\{(x_0 - 1/(N+1))/2, (1/N - x_0)/2\} > 0$, if $|x - x_0| \le \delta$ then $1/(N+1) < x < 1/N$. In all three cases we are showing that x_0 is not an adherent point unless it belongs to $A_3 \cup \{0\}$, therefore the closure of A_3 is $A_3 \cup \{0\}$.

(d) The set A_4 is neither bounded above nor bounded below. Given $M \in \mathbb{R}$ by Archimedean property there is $N \in \mathbb{N}$ such that $M < N$, so no M can be an upper bound for $\mathbb{Z}$. Likewise, no $L \in \mathbb{R}$ can be a lower bound for $\mathbb{Z}$.

The set A_4 doesn't have a least upper bound or a greatest lower bound. In the extended real numbers we have $\sup A_4 = \infty$ (this means precisely that the set is not bounded above) and $\inf A_4 = -\infty$ (this means precisely that the set is not bounded below).

The set A_4 is closed therefore is equal to its closure $\overline{A_4} = A_4$. Given any $x_0 \in \mathbb{R}\backslash\mathbb{Z}$, there is unique $N \in \mathbb{Z}$ (by interspersing reals by integers) such that $N < x_0 < N + 1$, let $\delta = \min\{(x_0 - N)/2, (N +$

[4]Because if $x_0 \in A$ then given any $\delta > 0$ there is $x = x_0 \in A$ such that $|x - x_0| = |x_0 - x_0| = 0 \le \delta$, in other words x_0 is an adherent point of A.

[5]Apply interspersing of positive reals by naturals to $y_0 = 1/x_0 > 1$, there is unique $1 \le N \in \mathbb{N}$ such that $N \le y_0 = 1/x_0 < N + 1$, take reciprocals, and since x_0 is not in A_3 get strict inequalities.

$1-x_0)/2\} > 0$, if $|x-x_0| \le \delta$ then $N < x < N+1$, that is x is not in $\mathbb{Z}$. This means that if $x_0 \in \mathbb{R}\backslash\mathbb{Z}$ then it is NOT adherent to $\mathbb{Z}$, so $\overline{\mathbb{Z}} = \mathbb{Z}$.

4. For each choice of subsets A_i of the real numbers in Exercise 2, construct a function $f_i : \mathbb{R} \to \mathbb{R}$ that has discontinuities at every point $x \in A_i$ and is continuous on its complement $\mathbb{R} \setminus A_i$. Explain.

SOLUTION:

(a) Let $f_1(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0,1] \\ 0 & \text{otherwise} \end{cases}$

f_1 is discontinuous at every point $x \in [0,1]$ because there are rationals $0 \le p_n \le 1$ converging to x and there are irrationals $0 \le q_n \le 1$ converging to x (why?[6]), so that $\lim_{n\to\infty} f_1(p_n) = 1 \neq 0 = \lim_{n\to\infty} f_1(q_n)$.

f_1 is continuous at every point $x_0 < 0$ or $x_0 > 1$. Because for such an x_0 there is $\delta > 0$ (just take $\delta = |x_0|/2 > 0$ when $x_0 < 0$, or $\delta = (x_0 - 1)/2 > 0$ when $x_0 > 1$), such that if $|x - x_0| \le \delta$ then x is not in $A_1 = [0,1]$. Therefore $|f_1(x) - f_1(x_0)| = |0 - 0| = 0 \le \epsilon$ for all $|x - x_0| \le \delta$ and for all $\epsilon > 0$. Thus f_1 is continuous at x_0 not in $A_1 = [0,1]$, and discontinuous at every point in A_1 as requested.

(b) Let $f_2(x) := x f_1(x)$, then f_2 is still continuous for all $x < 0$ and for all $x > 1$ (why?[7]), is still discontinuous for all $0 < x \le 1$ by similar argument to the one we used for f_1 (really?[8]). It remains to be checked that f_2 is continuous at $x = 0$. But that we deduce from the squeeze theorem as $x \to 0$ since $0 \le f_2(x) = x f_1(x) \le x$.

(c) Let f_3 be an infinite staircase with steps on the intervals $[\frac{1}{n+1}, \frac{1}{n})$, steps decreasing in height to zero as n goes to infinity. More precisely:

$$f_3(x) = \begin{cases} 0 & \text{if } x \le 0 \\ \frac{1}{n+1} & \text{if } x \in [\frac{1}{n+1}, \frac{1}{n}) \\ 1 & \text{if } x \ge 1 \end{cases}$$

f_3 is discontinuous at $1/n$ for each $n > 0$, $n \in \mathbb{N}$. Simply notice that the right and left limits are different (coming from different steps). If $x \ge 1$ then $f_3(x) = 1$ so $\lim_{x\to 1^+} f_3(x) = 1 = f_3(1)$, however when $1/2 \le x < 1$ then $f_3(x) = 1/2$ so $\lim_{x\to 1^-} f_3(x) = 1/2 \neq f(1)$, so f_3 is not continuous at $x = 1$. When $n \ge 2$ and $1/n \le x < 1/(n-1)$ then $f_3(x) = 1/n$ so that:

$$\lim_{x\to(1/n)^+} f_3(x) = \lim_{x\to(1/n)^+} \frac{1}{n} = \frac{1}{n} = f_3(1/n).$$

Similarly, when $1/(n+1) < x < 1/n$ then $f_3(x) = 1/(n+1)$, therefore

$$\lim_{x\to(1/n)^-} f_3(x) = \lim_{x\to(1/n)^-} \frac{1}{n+1} = \frac{1}{n+1} \neq f_3(1/n).$$

If $x_0 < 0$ (or $1/(n+1) < x_0 < 1/n$, or $x_0 > 1$) then one can find $\delta = |x_0|/2 > 0$ (or $\delta = \frac{1}{2}\min\{x_0 - \frac{1}{n+1}, \frac{1}{n} - x_0\}$, or $\delta = \frac{x_0 - 1}{2}$) so that if $|x - x_0| \le \delta$ then $x < 0$ (or $1/(n+1) < x < 1/n$ or $x > 1$). In all three cases, for the appropriate $\delta > 0$, if $|x - x_0| \le \delta$ then $f_3(x) = f_3(x_0)$ because x, x_0 are both on the same step. That is for all $\epsilon > 0$ there is $\delta > 0$ so that $|f_3(x) - f_3(x_0)| = 0 \le \epsilon$ for all $|x - x_0| \le \delta$, thus f_3 is continuous at such x_0.

If $x_0 = 0$ then for all $\epsilon > 0$ there is $N > 0$ such that $1/N < \epsilon$ (Archimedian Principle), let $\delta := 1/N > 0$, and by definition if $|x| \le 1/N$ then $|f_3(x)| \le 1/N < \epsilon$. That is f_3 is continuous at $x_0 = 0$ since $f_3(0) = 0$.

[6] Take your favorite irrational positive number, say $\sqrt{2}$, and given $\delta > 0$ there is $N > 0$ by the Arquimedian property such that $\sqrt{2}/N \le \delta/2$. The numbers $m\sqrt{2}/n$ with $m, n \in \mathbb{Z}$ are all irrationals (otherwise $\sqrt{2}$ will be rational) and if x, y are real numbers and $|x - y| = \delta$ there is $m \in \mathbb{Z}$ such that $m\sqrt{2}/N$ is in between x and y.

[7] Because f_2 is the product of two continuous functions ($g(x) = x$ and $f_1(x)$) at such points x.

[8] Except that this time the images of the sequence of rationals p_n converging to x will be p_n instead of 1, so that $\lim_{n\to\infty} f_2(p_n) = \lim_{n\to\infty} p_n = x$, and the images of the irrationals q_n converging to x will still be zero, but since $x \neq 0$. We conclude that f_2 is not continuous at x, for any $0 < x \le 1$.

(d) Let f_4 be another infinite staircase defined by $f_4(x) = n$ for all $n \leq x < n+1$ and $n \in \mathbb{Z}$. This function has jump discontinuities at each integer, indeed, $\lim_{x\to n^-} f_4(x) = n-1 \neq n = \lim_{x\to n^+} f_4(x)$. If $x_0 \in \mathbb{R}\backslash\mathbb{Z}$ then we can find $N \in \mathbb{Z}$ such that $N < x_0 < N+1$ by interspersing of reals by integers and if we let $\delta = \frac{1}{2}\min\{x_0 - N, N+1-x_0\} > 0$, then for all $x \in \mathbb{R}$ such that $|x - x_0| \leq \delta$ we have $N < x < N+1$ (why?[9]), so $f_4(x) = N$ therefore $|f_4(x) - f_4(x_0)| = |N - N| = 0 \leq \epsilon$ for all $\epsilon > 0$. This means f_4 is continuous at all points in $\mathbb{R}\backslash\mathbb{Z}$.

5. Let $f : [0,1] \to [0,1]$ be a continuous function. Show that there exists a real number x in $[0,1]$ such that $f(x) = x$, a "fixed point" (Exercise 9.7.2 p.241-242 2nd edition).

SOLUTION: Consider the function $g(x) = f(x) - x$, g is the difference of two continuous functions, f and the identity function $h(x) = x$, hence g is continuous on $[0,1]$. Evaluating g at the endpoints we observe that $g(0) = f(0)$ by hypothesis on f, $0 \leq g(0) \leq 1$; similarly, $g(1) = f(1) - 1$ and since $0 \leq f(x) \leq 1$ for all $x \in [0,1]$ then $-1 \leq g(1) \leq 0$. If $g(0) = 0$ then $f(0) = 0$ and we can set the fixed point to be $x = 0$. If $g(1) = 0$ then $f(1) = 1$ and we can set the fixed point to be $x = 1$. Assume now that $g(0) \neq 0$ and $g(1) \neq 0$ then $g(1) < 0 < g(0)$, and since g is continuous on $[0,1]$, by the intermediate value theorem, there must be a point $c \in [0,1]$ such that $g(c) = 0$, but then $f(c) = c$ and we can set the fixed point to be $x = c$.

6. Let $a < b$ be real numbers, and let $f : [a,b] \to \mathbb{R}$ be a function which is both continuous and one-to-one. Show that f is strictly monotone. (See hint in Exercise 9.8.3 p. 241 2nd ed).

SOLUTION: Since f is continuous on $[a,b]$ it will reach its maximum and minimum somewhere in the interval $[a,b]$, that is there are points $a \leq x_{min}, x_{max} \leq b$ such that

$$f(x_{min}) \leq f(x) \leq f(x_{max}), \quad \text{for all } x \in [a,b].$$

If x_{min} is in the open interval (a,b) then both $f(a)$ and $f(b)$ must be strictly larger than $f(x_{min})$ (otherwise there will be two different points mapped to the same value violating the fact that f was assumed to be one-to-one), and either $f(x_{min}) < f(b) < f(a)$ or $f(x_{min}) < f(a) < f(b)$ (equality is forbidden beacuse of the injectivity of f). In the first case, by intermediate value theorem applied to the continuous function f on the interval $[a, x_{mim}]$, there will be a point x such that $a < x < x_{min} < b$ and $f(x) = f(b)$, which is not possible because f is one-to-one. In the second case there is a point x such that $a < x_{min} < x < b$ and $f(x) = f(a)$ which is impossible for the same reason. We conclude that x_{min} must be a or b, and by a similar argument that x_{max} must be b or a.

Case 1: $f(a) < f(x) < f(b)$ for all $x \in (a,b)$ (that is $x_{min} = a$ and $x_{max} = b$). In this case f must be strictly increasing. Suppose not, then there are $a \leq x < y \leq b$ such that $f(x) \geq f(y)$, equality can not happen because of injectivity, so that in fact $f(x) > f(y) > f(a)$. Apply now the Intermediate Value Theorem to the continuous function f on the interval $[a,x]$, there is a point $a \leq z \leq x$ such that $f(z) = f(y)$, but f is one-to-one therefore $z = y$, and $z \leq x < y$ so that $z \neq y$, we have reached a contradiction, therefore in this case, f must be strictly increasing.

Case 2: $f(b) < f(x) < f(a)$ for all $x \in (a,b)$ (that is $x_{min} = b$ and $x_{max} = a$). A similar argument shows that in this case f must be strictly decreasing.

[9] We have $N < x_0 < N+1$, $-(x_0 - N)/2 \leq -\delta$ and $\delta \leq (N+1-x_0)/2$, and we have $x_0 - \delta \leq x \leq x_0 + \delta$, together we get

$$N < \frac{N}{2} + \frac{N}{2} < \frac{x_0}{2} + \frac{N}{2} = x_0 - \frac{x_0 - N}{2} \leq x_0 - \delta \leq x \leq x_0 + \delta \leq x_0 + \frac{N+1-x_0}{2} < \frac{x_0}{2} + \frac{N+1}{2} < \frac{N+1}{2} + \frac{N+1}{2} = N+1,$$

thus $N < x < N+1$.

7. LET $f : \mathbb{R} \to \mathbb{R}$ THAT SATISFIES THE MULTIPLICATIVE PROPERTY $f(x+y) = f(x)f(y)$ FOR ALL $x, y \in \mathbb{R}$. ASSUME f IS NOT IDENTICALLY EQUAL TO ZERO.

(i) SHOW THAT $f(0) = 1$, $f(x) \neq 0$ FOR ALL $x \in \mathbb{R}$, AND $f(-x) = \frac{1}{f(x)}$ FOR ALL $x \in \mathbb{R}$. SHOW THAT $f(x) > 0$ FOR ALL $x \in \mathbb{R}$.

SOLUTION: By hypothesis there is $x_0 \in \mathbb{R}$ such that $f(x_0) \neq 0$, apply multiplicative property to $x_0 = 0 + x_0$ to get $f(x_0) = f(0 + x_0) = f(0)f(x_0)$, use multiplicative cancellation (since $f(x_0) \neq 0$) to conclude $f(0) = 1$.

Assume there is $x_0 \in \mathbb{R}$ such that $f(x_0) = 0$ then f is identically equal to zero contradicting the fact that it is not. More precisely, given $x \in \mathbb{R}$ then using the multiplicative property we get $f(x) = f(x_0 + (x - x_0)) = f(x_0)f(x - x_0) = 0$. We conclude that $f(x) \neq 0$ for all $x \in \mathbb{R}$.

Apply multiplicative property again to get $1 = f(0) = f(x + (-x)) = f(x)f(-x)$ (which also implies $f(x) \neq 0$) for all $x \in \mathbb{R}$ and since $f(x) \neq 0$ then $f(-x) = 1/f(x)$.

Finally, since for all $x \in \mathbb{R}$ we have $x = x/2 + x/2$ then $f(x) = f(x/2)f(x/2) = f(x/2)^2 > 0$, since $f(x/2) \neq 0$.

(ii) LET $a = f(1)$ (BY (I) $a > 0$). SHOW THAT $f(n) = a^n$ FOR ALL $n \in \mathbb{N}$. USE (i) TO SHOW THAT $f(z) = a^z$ FOR ALL $z \in \mathbb{Z}$.

SOLUTION: By definition $f(1) = a$, we will show that $f(n) = a^n$ for all $n \in \mathbb{N}$ by induction.

Base case: $n = 0$, we showed in part (I) that $f(0) = 1 = a^0$.

Inductive step: assume $f(n) = a^n$ show that $f(n+1) = a^{n+1}$. First by multiplicative formula, then by inductive hypothesis we get

$$f(n+1) = f(n)f(1) = a^n a = a^{n+1}.$$

By part (i) for all $n \in \mathbb{N}$, $f(-n) = 1/f(n) = 1/a^n = a^{-n}$. Together we conclude that for all $z \in \mathbb{Z}$ $f(z) = a^z$.

(iii) SHOW THAT $f(r) = a^r$ FOR ALL $r \in \mathbb{Q}$.

SOLUTION: First note we can show by induction that the following multiplicative property for n summands holds:

$$f(x_1 + x_2 + \cdots + x_n) = f(x_1)f(x_2)\ldots f(x_n). \tag{1}$$

Base case: is $n = 2$ the given multiplicative property.

Inductive step: assume (1) show that $f(x_1 + x_2 + \cdots + x_n + x_{n+1}) = f(x_1)f(x_2)\ldots f(x_n)f(x_n + 1)$. Now using first associative property for addition, then the multiplicative property for two summands, then the inductive hypothesis we get want we want:

$$f((x_1 + x_2 + \cdots + x_n) + x_{n+1}) = f(x_1 + x_2 + \cdots + x_n)f(x_{n+1}) = f(x_1)f(x_2)\ldots f(x_n)f(x_n + 1).$$

Note that $1 = n/n = 1/n + \cdots + 1/n$ where we added n times $1/n$, then by the multiplicative property for n summands we get $a = f(1) = f(1/n)^n$, taking nth root $(a > 0)$ we get, $f(1/n) = a^{1/n}$.

Next we observe that for all $m > 0$, $m/n = 1/n + \cdots + 1/n$ where we have added m times $1/n$, by multiplicative property we get $f(m/n) = f(1/n)^m = (a^{1/n})^m = a^{m/n}$. Finally by part (I), $f(-m/n) = 1/f(m/n) = 1/a^{m/n} = a^{-m/n}$. We have shown that for all rationals $r \in \mathbb{Q}$, $f(r) = a^r$.

(iv) SHOW THAT IF f IS CONTINUOUS AT $x = 0$, THEN f IS CONTINUOUS AT EVERY POINT IN $\mathbb{R}$. MOREOVER $f(x) = a^x$ FOR ALL $x \in \mathbb{R}$.

SOLUTION: We are given that f is continuous at $x_0 = 0$. This means that $\lim_{x\to 0} f(x) = f(0) = 1$. We need to show that f is continuous at any $x_0 \in \mathbb{R}$, that is we need to show that $\lim_{x\to x_0} f(x) = f(x_0)$.

By multiplicative property, $f(x) = f(x_0 + (x - x_0)) = f(x_0)f(x - x_0)$ and if $x \to x_0$ then $(x - x_0) \to 0$, substituting in the limit, and using the continuity of f at zero, we get

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} f(x_0)f(x - x_0) = f(x_0) \lim_{(x - x_0) \to 0} f(x - x_0) = f(x_0)f(0) = f(x_0).$$

Now we know that under the assumption of continuity at 0, f is continuous everywhere on $\mathbb{R}$. In particular given any real number x there is a sequence of rationals $\{p_n\}_{n \geq 0}$ that converges to x, hence by continuity their images will converge to $f(x)$, that is $\lim_{n \to \infty} f(p_n) = f(x)$. But we proved in part (iii) that $f(p_n) = a^{p_n}$ for any $p_n \in \mathbb{Q}$, and the exponential is continuous by definition, so $\lim_{n \to \infty} f(p_n) = \lim_{n \to \infty} a^{p_n} = a^{\lim_{n \to \infty} p_n} = a^x$. Finally, since limits are unique we conclude that $f(x) = a^x$ for all $x \in \mathbb{R}$.

8. Decide whether the functions $f_i : X_i \to \mathbb{R}$ are uniformly continuous or not on their domain.

(a) $f_1(x) = x^{13} - 8x^5 + 7 + 2^x$ with $X_1 = [-3, 14]$,

(b) $f_2(x) = x^2$ with $X_2 = [1, \infty)$,

(c) $f_3(x) = 1/x$ with $X_3 = (0, 2]$,

(d) $f_4(x) = \sqrt{x}$ with $X_4 = [0, \infty)$,

SOLUTION:

(a) f_1 is a continuous function since it it the sum of two continuous functions: a polynomial $x^{13} - 8x^5 + 7$ and and exponential function 2^x. Continuous functions on a closed and bounded interval (such as $[-3, 14]$) are uniformly continuous. Hence f_1 is uniformly continuous.

(b) f_2 is a continuous function being a monomial. However it fails to be uniformly continuous and the problem arises as x goes to infinity. Consider the sequences $x_n = n$ and $y_n = n + 1/n$, these are equivalent sequences (why?[10]) however their images $f_2(x_n) = n^2$ and $f_2(y_n) = n^2 + 2 + 1/n^2$ are NOT equivalent sequences (why?[11]). Since uniformly continuous functions take equivalent sequences to equivalent sequences this shows that f_2 is not uniformly continuous on X_2.

(c) f_3 is continuous on its domain since it is the reciprocal of the continuous function x that never vanishes on the domain X_3. However f_3 is not uniformly continuous and the problem arises as x goes to zero. Consider the Cauchy sequence $x_n = 1/n$, its image $f_3(x_n) = n$ is NOT a Cauchy sequence (why?[12]). Since uniformly continuous functions take Cauchy sequences to Cauchy sequences this shows that f_3 is not uniformly continuous on X_3.

(d) f_4 is continuous on $[0, \infty)$ since it is a positive power $f_4(x) = x^{1/2}$. Moreover, f_4 is uniformly continuous on any closed and bounded interval of the form $[0, b]$ (notice that the same can be said about f_2 which is NOT uniformly continuous on $[0, \infty)$), but in this case the slow growth of the function at infinity ensures that f_4 is uniformly continuous on X_4. More precisely, if $x, y \geq 1$ then $|\sqrt{x} - \sqrt{y}| \leq |x - y|$ (why?[13]), so given $\epsilon > 0$ let $\delta = \epsilon > 0$ if $x, y \geq 1$ then $|\sqrt{x} - \sqrt{y}| \leq \delta = \epsilon$ so f_4 is uniformly continuous on $[1, \infty)$. Now use Part (a) in Exercise 10 with intervals determined by $a = 0$, $b = 1$, and $c = \infty$ to deduce that f_4 is uniformly continuous on $[0, \infty)$.

9. Show that if $f : [a, b] \to \mathbb{R}$ is continuous then it is uniformly continuous.

SOLUTION: We will proceed by contradiction, assume f is not uniformly continuous this means (negating the sequential characterization of uniform continuity) that there are two equivalent sequences $\{x_n\} \sim \{y_n\}$ in $[a, b]$ whose images $\{f(x_n)\}$ and $\{f(y_n)\}$ are not equivalent sequences. In particular this means that there is an $\epsilon > 0$ such that for each $N > 0$ there is $n_N \geq N$ such that $|f(x_{n_N}) - f(y_{n_N})| > \epsilon$. We can always find among the labels $\{n_N\}_{N \geq 1}$ a strictly increasing sequence of labels $n_{N_1} < n_{N_2} <$

[10]Because $|x_n - y_n| = 1/n \to 0$ as $n \to \infty$.

[11]Because $|f_2(x_n) - f_2(y_n)| = 2 + 1/n^2 \geq 2$ for all $n \in \mathbb{N}$.

[12]Because $|f_3(x_n) - f_3(x_m)| = |m - n| \geq 1$ for all $n \neq m$, $n, m \in \mathbb{N}$.

[13]Because if $x, y \geq 1$ then $\sqrt{x}, \sqrt{y} \geq 1$ and $\frac{1}{\sqrt{x}+\sqrt{y}} \leq \frac{1}{1+1} = \frac{1}{2}$. Finally

$$|\sqrt{x} - \sqrt{y}| = \frac{|\sqrt{x} - \sqrt{y}||\sqrt{x} + \sqrt{y}|}{|\sqrt{x} - \sqrt{y}|} = \frac{|x - y|}{|\sqrt{x} + \sqrt{y}|} \leq \frac{|x - y|}{2} < |x - y|.$$

$n_{N_3} < \cdots < n_{N_k} < n_{N_{k+1}} < \dots$ (why?[14]) such that $|f(x_{n_{N_k}}) - f(y_{n_{N_k}})| > \epsilon$. As a consequence, we can assume, if necessary relabeling our sequence, that $\{x_{n_N}\}_{N\geq 1}$ and $\{y_{n_N}\}_{N\geq 1}$ are sequences in the closed and bounded set $[a,b]$. So by Exercise 1(b) (Heine-Borel Theorem) there is a convergent subsequence $\{x_{n_{N_j}}\}_{j\geq 1}$ to a point $x_0 \in [a,b]$. Since f is continuous on $[a,b]$ it is continuous at x_0 and therefore $\lim_{j\to\infty} f(x_{n_{N_j}}) = f(x_0)$. On the other hand the sequences $\{x_n\}$ and $\{y_n\}$ are equivalent which means that $\lim_{n\to\infty}(x_n - y_n) = 0$, and the same must hold for any subsequences, namely $\lim_{n\to\infty}(x_{n_{N_j}} - y_{n_{N_j}}) = 0$. Since $\lim_{n\to\infty} x_{n_{N_j}} = x_0$ then the limit laws imply $\lim_{n\to\infty} y_{n_{N_j}} = x_0$. Now continuity of f implies that $\lim_{j\to\infty} f(y_{n_{N_j}}) = f(x_0)$. Once again limit laws imply now that $\lim_{j\to\infty}(f(x_{n_{N_j}}) - f(y_{n_{N_j}})) = 0$ but on the other hand $|f(x_{n_{N_j}} - f(y_{n_{N_j}})| > \epsilon$ for all $j \geq 1$, this is a contradiction. It must be that the function f is uniformly continuous on $[a,b]$.

10. **(a)** Assume g is defined on an open interval (a,c) and it is known to be uniformly continuous on $(a,b]$ and on $[b,c)$ where $a < b < c$. Prove that g is uniformly continuous on (a,c).

SOLUTION: Given $\epsilon > 0$, by hypothesis for $\epsilon_0 = \epsilon/2 > 0$ there are $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$|f(x_1) - f(x_2)| \leq \epsilon_0, \quad \text{if } a < x_1, x_2 \leq b \text{ and } |x_1 - x_2| \leq \delta_1, \tag{2}$$

$$|f(y_1) - f(y_2)| \leq \epsilon_0, \quad \text{if } b \leq y_1, y_2 < c \text{ and } |y_1 - y_2| \leq \delta_2. \tag{3}$$

Let $\delta = \min\{\delta_1, \delta_2\} > 0$, and assume $a < z_1, z_2 < c$ and $|z_1 - z_2| \leq \delta$. Three configurations are possible:

1) $a < z_1, z_2 \leq b$ in which case let $x_1 = z_1$ and $x_2 = z_2$ in (2), since $\delta \leq \delta_1$, then $|f(z_1) - f(z_2)| \leq \epsilon_0 < \epsilon$.

2) $b \leq z_1, z_2 < c$ in which case let $y_1 = z_1$ and $y_2 = z_2$ in (3), since $\delta \leq \delta_2$, then $|f(z_1) - f(z_2)| \leq \epsilon_0 < \epsilon$.

3) $a < z_1 < b < z_2 < c$, then $|b - z_1| \leq \delta$ and $|z_2 - b| \leq \delta$, and can use respectively (2) with $x_1 = z_1$, $x_2 = b$, and (3) with $y_1 = b$, $y_2 = z_2$, to conclude that $|f(z_2) - f(b)| \leq \epsilon_0$ and $|f(b) - f(z_1)| \leq \epsilon_0$. Finally using the triangle inequality we get that:

$$|f(z_2) - f(z_1)| \leq |f(z_2) - f(b)| + |f(b) - f(z_1)| \leq 2\epsilon_0 = \epsilon.$$

3') the case $a < z_2 < b < z_1 < b$ is exactly the same as case 3).

In all cases as long as $|z_1 - z_2| \leq \delta$ we conclude that $|f(z_2) - f(z_1)| \leq \epsilon$. That is, f is uniformly continuous on the interval (a,c).

Note that the interval (a,c) is not required to be bounded, it is possible for $a = -\infty$ or $c = +\infty$ (or both).

(b) Show that if f is uniformly continuous on (a,b) and (b,c), for some $b \in (a,c)$, then f is uniformly continuous on (a,c) if and only if f is continuous at b.

SOLUTION: [⇒] If f is uniformly continuous on (a,c) then it is continuous at b.

[⇐] Assume now that f is continuous at b and it is uniformly continuous on the smaller intervals (a,b) and (b,c). If we can show that f is uniformly continuous on $(a,b]$ and on $[b,c)$ then by part (a) we will conclude that f is uniformly continuous on (a,c) as required.

We will show that under the hypothesis f is uniformly continuous on $(a,b]$, a similar argument will take care of $[b,c)$. Given $\epsilon > 0$ there are $\delta_1 > 0$ and $\delta_2 > 0$ such that if $|x - b| \leq \delta_1$ then $|f(x) - f(b)| \leq \epsilon$ (continuity of f at b), and if $a < x, y < b$ and $|x - y| \leq \delta_2$ then $|f(x) - f(y)| \leq \epsilon$ (uniform continuity of f on (a,b)). Given $\epsilon > 0$ let $\delta = \min\{\delta_1, \delta_2\} > 0$, then if $a < x, y \leq b$ and $|x - y| \leq \delta$ then either $x, y < b$ or one of them coincides with b, for example $y = b$, in both cases $|f(x) - f(y)| \leq \epsilon$. Therefore f is uniformly continuous on $(a,b]$.

[14] Let $N_1 = 1$, define recursively N_k for $k > 1$, given N_k such that $n_{N_{k-1}} < n_{N_k}$ let $N_{k+1} := n_{N_k} + 1$ then by construction $n_{N_{k+1}} \geq N_{k+1} > n_{N_k}$.

11. LET $f : \mathbb{R} \to \mathbb{R}$ BE A DIFFERENTIABLE FUNCTION SUCH THAT f' IS BOUNDED. SHOW THAT f IS UNIFORMLY CONTINUOUS.

SOLUTION: Given $\epsilon > 0$ we must find $\delta > 0$ such that for all $x, y \in \mathbb{R}$ with $|x - y| \leq \delta$ then $|f(x) - f(y)| \leq \epsilon$.

Since f is assumed differentiable on $\mathbb{R}$ it is continuous on $\mathbb{R}$, and the Mean Value Theorem holds on any closed and bounded interval. Assume $x < y$ consider the interval $[x, y]$ (if $y < x$ then consider the interval $[y, x]$), then by the Mean Value Theorem there is a point c in between x and y such that $f'(c) = \frac{f(x)-f(y)}{x-y}$, but we are also told the derivative is bounded say by $M > 0$, we get

$$|f(x) - f(y)| = |f'(c)|\,|x - y| \leq M|x - y|. \tag{4}$$

Given $\epsilon > 0$ let $\delta = \epsilon/M > 0$, if $|x - y| \leq \delta$ then $|f(x) - f(y)| \leq M\delta = M\epsilon/M = \epsilon$. We have shown f is uniformly continuous. (Notice that (4) is saying that such functions satisfy a Lipschitz condition introduced in Problem 13).

12. VERIFY THE CHAIN RULE (SEE EXERCISE 10.1.7. P. 256 2ND ED).

SOLUTION: We are given $X, Y \subset \mathbb{R}$, functions $f : X \to Y$, $g : Y \to \mathbb{R}$, a limit point $x_0 \in X$, so that $y_0 = f(x_0) \in Y$ is a limit point in Y. Assume f is differentiable at x_0, and g is differentiable at y_0, we want to show that $g \circ f$ is differentiable at x_0 and furthermore $(g \circ f)'(x_0) = g'(f(x_0))f'(x_0)$.

We will show both statements in one stroke by showing that

$$\lim_{x \to x_0} \frac{(g \circ f)(x) - (g \circ f)(x_0)}{x - x_0} = g'(f(x_0))f'(x_0).$$

We will use Newton's approximation theorem, that is we will show that for every $\epsilon > 0$ there is a $\delta > 0$ such that if $|x - x_0| \leq \delta$ then

$$|(g \circ f)(x) - (g \circ f)(x_0) - g'(f(x_0))f'(x_0)(x - x_0)| \leq \epsilon|x - x_0|. \tag{5}$$

Because f, g are differentiable we do have Newton's criterium for them.

(i) For every $\epsilon_1 > 0$ there is a $\delta_1 > 0$ such that if $|x - x_0| \leq \delta_1$ then

$$|f(x) - f(x_0) - f'(x_0)(x - x_0)| \leq \epsilon_1|x - x_0|.$$

(ii) For every $\epsilon_2 > 0$ there is a $\delta_2 > 0$ such that if $|y - y_0| \leq \delta_2$ then

$$|g(y) - g(y_0) - g'(y_0)(y - y_0)| \leq \epsilon_2|y - y_0|.$$

Note that we may assume $|g'(y_0)| > 0$ otherwise what we want is (ii).

Let $y = f(x)$ and we are given $y_0 = f(x_0)$, note that (i) implies that if $|x - x_0| \leq \delta_1$ then

$$(iii) \qquad |y - y_0| \leq (\epsilon_1 + |f'(x_0)|)|x - x_0|.$$

Given $\epsilon > 0$, take the left hand side of the inequality (5), add and subtract appropriate terms (namely $g'(y_0)(y - y_0)$), apply triangle inequality, and use (i) and (iii) with $\epsilon_1 := \min\{1, \frac{\epsilon}{2(1+|g'(y_0)|)}\} > 0$ and corresponding $\delta_1 > 0$, and use (ii) with $\epsilon_2 := \frac{\epsilon}{2(1+|f'(x_0)|)} > 0$ and corresponding $\delta_2 > 0$. With these parameters let $\delta := \min\{\delta_1, \frac{\delta_2}{\epsilon_1+|f'(x_0)|}\} > 0$. If $|x - x_0| \leq \delta$ then $|x - x_0| \leq \delta_1$, and we have the inequality in (i) with $\epsilon_1 > 0$; also $|x - x_0| \leq \frac{\delta_2}{\epsilon_1+|f'(x_0)|}$ which by (iii) gives $|y - y_0| \leq \delta_2$ and we have the inequality in (ii) with $\epsilon_2 > 0$, and we also have (iii) with $\epsilon_1 \leq 1$, that is $|y - y_0| \leq (1 + |f'(x_0)|)|x - x_0|$. All together this gives what we want:

$$\begin{aligned}
&|g(f(x)) - g(f(x_0)) - g'(f(x_0))f'(x_0)(x - x_0)| \\
&\quad \leq \quad |g(y) - g(y_0) - g'(y_0)(y - y_0)| + |g'(y_0)||f(x) - f(x_0) - f'(x_0)(x - x_0)| \\
&\quad \leq \quad \epsilon_2|y - y_0| + |g'(y_0)|\epsilon_1|x - x_0| \\
&\quad \leq \quad (\epsilon_2(1 + |f'(x_0)|) + \epsilon_1|g'(y_0)|)\,|x - x_0| \\
&\quad \leq \quad \left(\frac{\epsilon}{2} + \frac{\epsilon}{2}\right)|x - x_0| \;=\; \epsilon|x - x_0|.
\end{aligned}$$

13. A FUNCTION $f : \mathbb{R} \to \mathbb{R}$ SATISFIES A LIPSCHITZ CONDITION WITH CONSTANT $M > 0$ IF FOR ALL $x, y \in \mathbb{R}$,

$$|f(x) - f(y)| \le M|x - y|.$$

ASSUME $h, g : \mathbb{R} \to \mathbb{R}$ EACH SATISFY A LIPSCHITZ CONDITION WITH CONSTANT M_1 AND M_2 RESPECTIVELY.

(a) SHOW THAT $(h + g)$ SATISFIES A LIPSCHITZ CONDITION WITH CONSTANT $(M_1 + M_2)$.

SOLUTION: Use the triangle inequality and the Lipschitz conditions for h, g to get,

$$|(g+h)(x)-(g+h)(y)| = |(g(x)+h(x))-(g(y)+h(y))| \le |g(x)-g(y)|+|h(x)-h(y)| \le M_2|x-y|+M_1|x-y|.$$

Therefore $|(g + h)(x) - (g + h)(y)| \le (M_1 + M_2)|x - y|$ for all $x, y \in \mathbb{R}$, that is $(h + g)$ satisfies a Lipschitz condition with constant $(M_1 + M_2)$.

(b) SHOW THAT THE COMPOSITION $(h \circ g)$ SATISFIES A LIPSCHITZ CONDITION. WITH WHAT CONSTANT?

SOLUTION: Apply first the Lipschitz condition of h then the Lipschitz condition of g to get,

$$|h(g(x)) - h(g(y))| \le M_1|g(x) - g(y)| \le M_1M_2|x - y|.$$

Hence $h \circ g$ satisfies a Lipschitz condition with constant M_1M_2.

(c) SHOW THAT THE PRODUCT (hg) DOES NOT NECESSARILY SATISFY A LIPSCHITZ CONDITION. HOWEVER IF BOTH FUNCTIONS ARE BOUNDED THEN THE PRODUCT SATISFIES A LIPSCHITZ CONDITION.

SOLUTION: Consider $h(x) = g(x) = x$ these are Lipschitz functions with constant 1 (why?[15]). Suppose (hg) satisfies a Lipschitz condition, then there is an $M > 0$ such that $|(hg)(x) - (hg)(y)| = |x^2 - y^2| = |x - y||x + y| \le M|x - y|$ for all $x, y \in \mathbb{R}$. Let $x = M + 1$, $y = 0$ then we will conclude that $(M + 1) \le M$ which is false. Thus $(hg)(x) = x^2$ does not satisfy a Lipschitz condition. In this example neither h nor g are bounded functions.

Assume now that g, h are Lipschitz and bounded functions, so that there exists $B_1 > 0$ and $B_2 > 0$ such that $|h(x)| \le B_1$, and $|g(x)| \le B_2$ for all $x \in \mathbb{R}$. We will add and subtract $0 = g(x)h(y) - g(x)h(y)$ and use the triangle inequality to conclude that:

$$|g(x)h(x) - g(y)h(y)| \le |g(x)|\,|h(x) - h(y)| + |g(x) - g(y)|\,|h(y)| \le B_2M_1|x - y| + B_1M_2|x - y|.$$

Therefore $|g(x)h(x) - g(y)h(y)| \le (B_2M_1 + B_1M_2)|x - y|$ for all $x, y \in \mathbb{R}$. Hence hg satisfies a Lipschitz condition with constant $(B_2M_1 + B_1M_2)$.

14. ASSUME KNOWN THAT THE DERIVATIVE OF $f(x) = \sin x$ EQUALS $\cos x$, THAT IS, f IS DIFFERENTIABLE ON $\mathbb{R}$ AND $f'(x) = \cos x$. SHOW THAT $f : [0, \pi/2) \to [0, 1)$ IS INVERTIBLE, AND THAT ITS INVERSE $f^{-1} : [0, 1) \to [0, \pi/2)$ IS DIFFERENTIABLE. FIND THE DERIVATIVE OF THE INVERSE FUNCTION.

SOLUTION: Since f is differentiable on its domain then f is continuous. Moreover, for $0 \le x < \pi/2$, $f'(x) = \cos x > 0$, this implies that the function $f(x)$ is strictly increasing and since $f(\pi/2) = 1 > f(x)$ for all $0 \le x < \pi/2$, then f is strictly increasing and continuous on the closed interval $[0, \pi/2]$. Therefore $f : [0, \pi/2] \to [f(0), f(\pi/2)] = [0, 1]$ is invertible and its inverse $f^{-1} : [0, 1] \to [0, \pi/2]$ is continuous and strictly increasing by Proposition 9.8.3. Finally to apply the Inverse Function theorem we need not only that f is differentiable at x and f^{-1} is continuous at $f(x)$ we also need $f'(x) \ne 0$, that is why the endpoint $\pi/2$ has been removed, since $f'(x) = \cos x = 0$ for $x \in [0, \pi/2]$ iff $x = \pi/2$. For $f : [0, \pi/2) \to [0, 1)$ all the hypothesis of the Inverse Function theorem apply and we conclude that f^{-1} is differentiable, furthermore the formula for the derivative of the inverse holds for any $0 \le y < 1$:

$$(f^{-1})'(y) = \frac{1}{f(f^{-1}(y))} = \frac{1}{\cos\big(\arcsin(y)\big)} = \frac{1}{\sqrt{1 - y^2}}.$$

[15] The identity function $f(x) = x$ then $|f(x) - f(y)| = |x - y| \le |x - y|$ for all $x, y \in \mathbb{R}$. Therefore the identity function satisfies a Lipschitz condition with constant 1.

15. As in the previous exercise we know that the function $\sin x$ is differentiable, and you can use known properties such as $\lim_{x\to 0, x\neq 0} \frac{\sin x}{x} = 1$ if need be. Let $G : \mathbb{R} \to \mathbb{R}$ be defined by

$$G(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Show that G is differentiable on $\mathbb{R}$ but G' is not continuous at zero.

SOLUTION: The function G is continuous and differentiable at all $x \neq 0$ (why?[16]), the derivative at those points can be calculated using the product and the chain rule:

$$G'(x) = 2x\sin(1/x) + x^2\cos(1/x)(-1/x^2) = 2x\sin(1/x) - \cos(1/x) \qquad \text{if } x \neq 0.$$

At $x = 0$ we must calculate the derivate by definition,

$$\lim_{x\to 0} \frac{G(x) - G(0)}{x - 0} = \lim_{x\to 0} \frac{x^2 \sin(1/x)}{x} = \lim_{x\to 0} x\sin(1/x).$$

The limit on the right-hand-side exists and equals zero (here we are using that the function $\sin(1/x)$ is bounded by one and the squeeze theorem: since $-x \leq x\sin(1/x) \leq x$). We conclude that G is differentiable at zero and $G'(0) = 0$.

The derivative is not continuous at $x = 0$, because

$$\lim_{x\to 0} G'(x) = \lim_{x\to 0} \big(2x\sin(1/x) - \cos(1/x)\big),$$

and this limit does not exist (the first term does go to zero by the squeeze theorem, but the second oscillates wildly between ± 1).

16. Give an example of a function on $\mathbb{R}$ that has the intermediate value property for every interval (that is it takes on all values between $f(a)$ and $f(b)$ on $a \leq x \leq b$ for all $a < b$), but fails to be continuous at a point. Can such function have a jump discontinuity?

SOLUTION: $f(x) = \sin(1/x)$ (a cousin of G in the previous Exercise). The function f is continuous except at $x = 0$ where the function oscillates wildly between ± 1. On any interval that does not contain $x = 0$ the function is continuous and therefore it has the intermediate value property on that interval. On an interval $[a, b]$ such that $a < 0 < b$, for each value y in between $f(a)$ and $f(b)$ there will be infinitely many points $a < x < b$ such that $f(x) = y$, so that the intermediate value property holds there too. More precisely, we know that $\sin z$ is a continuous periodic function of period 2π. For any $-1 \leq y \leq 1$ there is $0 < z \leq 2\pi$ such that $\sin z = y$, periodicity implies now that $\sin(z + 2\pi n) = y$ for all $n \in \mathbb{Z}$. This means that $f(1/(z + 2\pi n)) = y$ for all $n \in \mathbb{Z}$. Now one can find $N > 0$ such that $x_n = 1/(z + 2\pi n) \in (a, b)$ for all $|n| > N$, $x_n \to 0$ as $n \to \infty$, and $f(x_n) = y$ for all $n \in \mathbb{Z}$.

If you prefer not to use the sine function, you can try to define a piecewise continuous linear function that mimics the behaviour of $\sin(1/x)$.

Such a function cannot have a jump discontinuity. Suppose on the contrary that f has a jump discontinuity at x_0 and it has the intermediate value property on every interval. Then both limits on the right and on the left of x_0 exist but they are different, say,

$$\lim_{x\to x_0^+} f(x) = A, \quad \lim_{x\to x_0^-} f(x) = B, \quad |B - A| > 0.$$

From definition of limit, given $\epsilon = |B - A|/3 > 0$ there are $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$|f(x) - A| \leq \epsilon \text{ if } \; x_0 < x \leq x_0 + \delta_1; \quad |f(x) - B| \leq \epsilon \text{ if } \; x_0 - \delta_1 \leq x < x_0.$$

[16] The function G is the product of functions $f(x) = x^2$ and $g(x) = \sin(h(x))$ where $h(x) = 1/x$. The function f is a polynomial hence continuous and differentiable in $\mathbb{R}$, the function g is the composition of the sine function, a continuous and differentiable function in $\mathbb{R}$, and the function h which is continuous and differentiable when $x \neq 0$. Continuity and differentiability at a point are preserved by composition and product. All together G is continuous and differentiable at $x \neq 0$.

Let $\delta = \min\{\delta_1, \delta_2\} > 0$, the function f does not have the intermediate value property on the interval $[x_0 - \delta, x_0 + \delta]$. Without loss of generality we can assume $A < B$ so that $\epsilon = (B - A)/3 > 0$, note that for all $x_0 - \delta \leq x < x_0$, $f(x) \geq B - \frac{B-A}{3}$, similarly, for all $x_0 < x \leq x_0 + \delta$, $f(x) \leq A + \frac{B-A}{3}$, therefore all values $A + \frac{B-A}{3} < y < B - \frac{B-A}{3}$ except perhaps one (if $f(x_0)$ is one of them) will never be reached by images of points on $[x_0 - \delta, x_0 + \delta]$. Therefore if f has the intermediate value property on all intervals then it cannot have jump discontinuities.

17. (L'Hopital's Rule). Let $f, g : X \to \mathbb{R}$, $x_0 \in X$ is a limit point of X such that $f(x_0) = g(x_0) = 0$, f, g are differentiable at x_0, and $g'(x_0) \neq 0$.

(i) Show that there is some $\delta > 0$ such that $g(x) \neq 0$ for all $x \in X \cap (x_0 - \delta, x_0 + \delta) \setminus \{x_0\}$.

Hint: Use Newton's approximation theorem.

SOLUTION: Let us proceed by contradiction. Suppose that for all $\delta > 0$ there is an $x_\delta \neq x_0$ such that $0 < |x - x_0| \leq \delta$ and $g(x_\delta) = 0$. Newton's approximation theorem says that for all $\epsilon > 0$ there is $\delta > 0$ such that for all $|x - x_0| \leq \delta$ then, since $g(x_0) = 0$, it follows that,

$$|g(x) - g(x_0) - g'(x_0)(x - x_0)| = |g(x) - g'(x_0)(x - x_0)| \leq \epsilon |x - x_0|.$$

In particular this holds for $x = x_\delta$, that is,

$$|g(x_\delta) - g'(x_0)(x_\delta - x_0)| = |g'(x_0)||x_\delta - x_0| \leq \epsilon |x_\delta - x_0|.$$

since $|x_\delta - x_0| > 0$ we can divide through to conclude that for all $\epsilon > 0$ $|g'(x_0)| \leq \epsilon$ but this is equivalent to $g'(x_0) = 0$ which is a contradiction. Therefore there is some $\delta > 0$ such that $g(x) \neq 0$ for all $x \in X \cap (x_0 - \delta, x_0 + \delta)$.

(ii) Show that $\displaystyle\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{f'(x_0)}{g'(x_0)}$.

SOLUTION: By part (i) it is safe to write $f(x)/g(x)$ provided $0 < |x - x_0| \leq \delta$. We are given that f and g are differentiable at x_0 and that $g'(x_0) \neq 0$ so both limits on the right exist and we can use limit laws, below all limits are taken over x such that $0 < |x - x_0| \leq \delta$,

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \frac{x - x_0}{g(x) - g(x_0)} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \lim_{x \to x_0} \frac{x - x_0}{g(x) - g(x_0)} = \frac{f'(x_0)}{g'(x_0)}.$$

(iii) Show that the following version of L'Hopital's Rule is not correct. Under the above hypothesis then,

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}.$$

Hint: Consider $f(x) = G(x)$ (as in exercise 10), and $g(x) = x$ at $x_0 = 0$.

SOLUTION: Consider $f(x) = G(x)$ in Exercise 15 and $g(x) = x$, then $f'(x)/g'(x) = G'(x)$ which we showed did not have a limit as $x \to 0$, so the limit on the right of (iii) doesn't exist. However G is differentiable at $x = 0$ with $f'(0) = G'(x) = 0$, and $g'(0) = 1$, and one can verify that

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{x^2 \sin(1/x)}{x} = \lim_{x \to x_0} x \sin(1/x) = 0 = \frac{f'(0)}{g'(0)}.$$

(The one before last equality by the squeeze theorem).

18. Let $f : [1, \infty] \to \mathbb{R}$ be a monotone decreasing non-negative function. Then the sum $\sum_{n=1}^{\infty} f(n)$ is convergent if and only if $\sup_{N>0} \int_1^N f(x)dx$ is finite.

Show by constructing counterexamples that if the hypothesis of monotone decreasing is replaced by Riemann integrable on intervals $[1, N]$ for all $N > 0$ then both directions of the if and only if above are false.

SOLUTION: This is the integral test for series. Note that the sequence $b_N = \int_1^N f(x)dx$ is an increasing sequence because $f(x) \geq 0$, and $b_{N+1} = b_N + \int_N^{N+1} f(x)dx$. Therefore $\sup_{N>0} b_N = \lim_{N\to\infty} b_N$ and it is either a non-negative real number or $+\infty$. Define $\int_1^\infty f(x)dx := \sup_{N>0} b_N$. With this understanding the statement reads: the series of non-negative terms $\sum_{n=1}^\infty f(n)$ and the "improper integral" $\int_1^\infty f(x)dx$ converge or diverge simultaneously when f is monotone decreasing and non-negative. We have used here the fact that monotone and bounded functions are Riemann integrable on closed and bounded intervals (Section 11.6).

Because the function f is monotone decreasing on an interval $[a, b]$ the minimum value is $f(b)$ and the maximum value is $f(a)$. Consider the partition $\mathcal{P}_N$ of the interval $[1, N]$ given by the natural numbers $1 \leq n \leq N$. On each subinterval $I_n := [n, n+1)$ of the partition (with the last one being $I_{N-1} := [N-1, N]$) then $\sup_{x\in I_n} f(x) = f(n)$ and $\inf_{x\in I_n} f(x) = f(n+1)$, therefore the lower and upper Riemann sums corresponding to $\mathcal{P}_N$ are given by $L(f, \mathcal{P}_N) = \sum_{n=1}^{N-1} f(n+1)$, and $U(f, \mathcal{P}_N) = \sum_{n=1}^{N-1} f(n)$. SInce f is Riemann integrable on $[1, N]$ then

$$L(f, \mathcal{P}_N) = \sum_{n=1}^{N-1} f(n+1) \leq \int_{[0,N]} f(x)dx \leq \sum_{n=1}^{N-1} f(n) = U(f, \mathcal{P}_N).$$

The sums are also increasing because the terms are non-negative, so we can take limits as $N \to \infty$ to conclude that

$$\sum_{n=1}^{\infty} f(n+1) = \sum_{n=2}^{\infty} f(n) \leq \int_0^\infty f(x)dx \leq \sum_{n=1}^{\infty} f(n),$$

understanding that some of these limits can be infinity. In fact if the series diverges then the left most limit is ∞ and it forces the integral to be ∞. Similarly if the integral is ∞ it will push the sum on the right-hand side to be ∞. So integral and series diverge to infinity simultaneously. As for convergence, if the series converges it means the partial sums are bounded and so are the finite integrals b_N, by monotone sequence theorem the $\lim_{N\to\infty} b_N$ exists. Similarly if the limit exists then the increasing sequence $\{b_N\}$ is bounded but so are the partial sums on the righ-hand-side hence it is a convergent series (starting at $n = 2$). Noting that $\sum_{n=1}^\infty f(n) = f(1) + \sum_{n=2}^\infty f(n)$ then the original series starting at $n = 1$ must be convergent. Integral and series converge simultaneously.

As for the examples consider first a function that takes values $f(x) = 1/n$ for $n \leq x \leq n + 1/n$ for all $1 \leq n \in \mathbb{N}$, and $f(x) = 0$ otherwise, then the series $\sum_{n=1}^\infty f(n) = \sum_{n=1}^\infty 1/n$ will be the divergent harmonic series, however the supremum of the integrals will correspond to the convergent p-series with $p = 2$, $\sum_{n=1}^\infty 1/n^2$. Second consider now a function that a function that takes values $f(x) = 1$ for all $x\mathbb{N}$, $f(x) = 0$ for all $x = n \in \mathbb{N}$, this time the series $\sum_{n=1}^\infty f(n) = 0$ however the $\int_1^N f(x)dx = N$ so the supremum over $N > 0$ is infinity.

Enjoy your holidays!!!!! It was a pleasure working with all of you!