MATH 401 Review 1 Spring 2004



(1) $P(n): 1+x+x^2+\cdots+x^n = \frac{1-x^{n+1}}{1-x}$ } IH

Base cases: $n=1$ $P(1)$: $1+x = \frac{1-x^2}{1-x}$ ✓.

$n=0$ $P(0)$ $1 = \frac{1-x}{1-x}$ ✓

Inductive step: $P(n) \Rightarrow P(n+1)$: $1+x+\cdots+x^n+x^{n+1} = \frac{1-x^{n+2}}{1-x}$

by IH $(P(n))$ $(1+x+\cdots+x^n)+x^{n+1} = \frac{1-x^{n+1}}{1-x} + x^{n+1}$

$= \frac{1-x^{n+1}+x^{n+1}(1-x)}{1-x} = \frac{1-x^{n+1}+x^{n+1}-x^{n+2}}{1-x}$

$= \frac{1-x^{n+2}}{1-x}$ which is what we wanted to show $P(n+1)$

by induction then $P(n)$ is true for all $n \in \mathbb{N}$. ▨

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(2) Given set $X$, $A \subseteq X$, $B \subseteq X$ show $X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$

This is one of Morgan Laws

Given $x \in X \setminus (A \cup B) \iff x \in X$ and $x \notin (A \cup B)$

$\iff x \in X$ and $(x \notin A$ and $x \notin B)$

$\iff x \in X$ and $(x \notin A$ and $x \notin B)$

$\iff (x \in X$ and $x \notin A)$ and $(x \in X$ and $x \notin B)$

$\iff x \in X \setminus A$ and $x \in X \setminus B$

$\iff x \in (X \setminus A) \cap (X \setminus B)$

since these are double implications we conclude going forward that $X \setminus (A \cup B) \subseteq (X \setminus A) \cap (X \setminus B)$ and going backwards that $(X \setminus A) \cap (X \setminus B) \subseteq X \setminus (A \cup B)$.

all together we conclude:

$$X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B),$$

which is what we wanted to prove.

(3) $r, q \in \mathbb{Q}$ show if $r \cdot q = 0$ Then $r = 0$ or $q = 0$

we will show The contrapositive: if $r \neq 0$ and $q \neq 0$ then $r \cdot q \neq 0$

~~assume~~ $r, q \in \mathbb{Q}$, $r, q \neq 0$ $\Rightarrow$ $r = \frac{a}{b}$, $q = \frac{c}{d}$ with $a, b, c, d$ ~~positive~~ nonzero integers.

$\Rightarrow$ $r \cdot q = \frac{ac}{bd}$, since we do know That in $\mathbb{Z}$ if $a, c$ are not zero Then $a \cdot c \neq 0$. (similarly $bd \neq 0$)

$\Rightarrow$ $rq \neq 0$. which is what we wanted to show.

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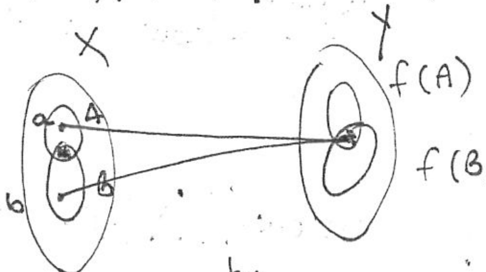
(4) $f: X \to Y$ $A, B \subseteq X$, $C, D \subseteq Y$

(a)




"it seems clear That $f(A \cap B) \subseteq f(A) \cap f(B)$ with equality when f is injetive"

if $y \in f(A \cap B)$ $\overset{\text{by def of image set}}{\Rightarrow}$ $\exists x \in A \cap B$ st $f(x) = y$

but $x \in A \cap B \Rightarrow x \in X$ and $x \in B$, and $f(x) = y$

$\overset{\text{by def of image set}}{\Rightarrow}$ $y \in f(A)$ and $y \in f(B)$

$\Rightarrow$ $y \in f(A) \cap f(B)$

$\therefore$ $f(A \cap B) \subseteq f(A) \cap f(B)$

If we try to walk backwards we will encounter an obstacle That can be overcome if we assume f injective

if $y \in f(A) \cap f(B)$ $\Rightarrow$ $y \in f(A)$ and $y \in f(B)$

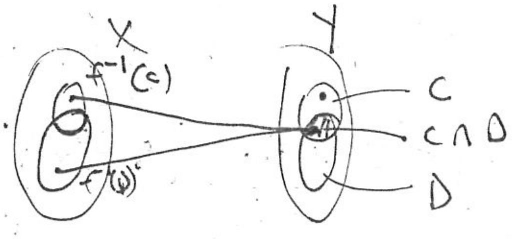
$\Rightarrow$ $\exists a \in A$ and $b \in B$ st $f(a) = y$ and $f(b) = y$

but unless we can ensure That There is $x \in A \cap B$ st $f(x) = y$ we cannot ensure That $y \in f(A \cap B)$.

If f were injective Then $f(a) = f(b) \Rightarrow a = b =: x$, so There is $x \in A \cap B$ st $f(x) = y$ $\Rightarrow$ $y \in f(A \cap B)$.

(4)

(b)




maybe $\exists y \in C \cap D$ that is not the image of anyone. Not that clear what is happening, let us try to see by working out details.

given $x \in f^{-1}(C \cap D) \xRightarrow{\text{by def of Inverse Image set}} y = f(x) \in C \cap D \xRightarrow{\text{def } \cap} f(x) \in C$ and $f(x) \in D$

$\xRightarrow{\text{by def of inverse image set}} x \in f^{-1}(C)$ and $x \in f^{-1}(D)$

$\xRightarrow{\text{def of } \cap} x \in f^{-1}(C) \cap f^{-1}(D)$ $\therefore$ $\boxed{f^{-1}(C \cap D) \subseteq f^{-1}(C) \cap f^{-1}(D)}$

This time all the steps are reversible since it is all definitions:

$x \in f^{-1}(C) \cap f^{-1}(D) \xRightarrow{\text{def } \cap} x \in f^{-1}(C)$ and $x \in f^{-1}(D)$

$\xRightarrow{\text{def}} f(x) \in C$ and $f(x) \in D \Rightarrow f(x) \in C \cap D$

$\xRightarrow{\text{Inverse Im. set}} x \in f^{-1}(C \cap D)$ $\therefore$ $\boxed{f^{-1}(C) \cap f^{-1}(D) \subseteq f^{-1}(C \cap D)}$

$\therefore$ $\boxed{f^{-1}(C) \cap f^{-1}(D) = f^{-1}(C \cap D)}$

(5) $A, B, C$ sets with $B \cap C = \phi$ Then

$$\#(A^B \times A^C) = \#(A^{B \cup C})$$

Recall: $A^B = \{ f : B \longrightarrow A,\ b \longrightarrow f(b) = a_1 \}$, $A^C = \{ g : C \longrightarrow A,\ c \longrightarrow g(c) = a_2 \}$

$A^B \times A^C = \{ (f, g) : f \in A^B$ and $g \in A^C \}$.

$A^{B \cup C} = \{ F : B \cup C \longrightarrow A,\ x \longrightarrow F(x) = a \}$

$x \in B \cup C \Rightarrow x \in B$ or $x \in C$ (exclusive or since $B \cap C = \phi$)

We need a bijection $\phi : A^B \times A^C \longrightarrow A^{B \cup C}$

$(f, g) \longrightarrow \phi(f, g) = F$

Define: $\phi(f, g)(x) = \begin{cases} f(x) & \text{if } x \in B \\ g(x) & \text{if } x \in C \end{cases}$ well defined since $B \cap C = \phi$.

$x \in B \cup C$

$\phi(f,g) : B \cup C \longrightarrow A$ by definition so
we most show is a bijection. $\phi : A^B \times A^C \longrightarrow A^{B \cup C}$

<u>one-to-one</u>: ~~suppose $x_1, x_2 \in B$ and~~

~~$\phi(f,g)(x_1) \neq \phi(f,g)(x_2)$~~ $\in A^B \times A^C$ and

Suppose $(f_1, g_1), (f_2, g_2) \in A^B \times A^C$ and

$\phi(f_1, g_1) = \phi(f_2, g_2)$

(Two functions are equal if and only if they have same domain and range and their images are identical)

$\Rightarrow \phi(f_1, g_1)(x) = \phi(f_2, g_2)(x)$ for all $x \in B \cup C$

$\Rightarrow$ if $x = b \in B$ then $\phi(f_1, g_1)(x) = f_1(x)$ $\Rightarrow$ $f_1(b) = f_2(b)$ $\forall b \in B$

$\phi(f_2, g_2)(x) = f_2(x)$ $\rightarrow f_1 = f_2$

Likewise

$\Rightarrow$ if $x = c \in C$ then $\phi(f_1, g_1)(x) = g_1(x)$ $\forall c \in C$

$\phi(f_2, g_2)(x) = g_2(x)$ $g_1(c) = g_2(c)$ $\Rightarrow g_1 = g_2$.

$\therefore (f_1, g_1) = (f_2, g_2)$ (by def of equality for ordered pairs)

$\therefore$ $\phi$ is injective or one-to one.

<u>Onto</u>: Given $F \in A^{B \cup C}$ then let $f(b) := F(b)$ $\forall b \in B$

$g(c) := F(c)$ $\forall c \in C$

$$\therefore \phi(f,g)(x) = \begin{cases} f(x) & x \in B \\ g(x) & x \in C \end{cases} = \begin{cases} F(x) & x \in B \\ F(x) & x \in C \end{cases} = F(x) \quad \forall x \in B \cup C.$$

$\therefore \phi(f,g) = F$ by def of equality of functions.

$\therefore$ $\phi$ is surjective or onto

$\therefore \#(A^B \times A^C) = \# A^{B \cup C}$ by definition.

⑥ $r \in \mathbb{Q}$, $n, m \in \mathbb{N}$. Define $r^0 := 1$, given $r^n \in \mathbb{Q}$ (5
define $r^{n+1} := r^n \times r \in \mathbb{Q}$ (in particular $0^0 := 1$)

(a) Show $(r^n)^m = r^{n \times m}$ by induction on $m$

$P(m): (r^n)^m = r^{n \times m}$

base case $m = 0$ $(r^n)^0 = 1$ by definition $\Rightarrow (r^n)^0 = r^{n \times 0}$ ✓.
$r^{n \times 0} = r^0 = 1$ by def.

Inductive Step: $P(m) \Rightarrow P(m+1)$: $(r^n)^{m+1} \overset{?}{=} r^{n \times (m+1)}$

by definition $(r^n)^{m+1} = (r^n)^m \times r^n \underset{P(m)}{\overset{(IH)}{=}} r^{n \times m} \cdot r^n$

assuming $= r^{n \times m + n} = r^{n \times (m+1)}$ ✓

$k, q \in \mathbb{N}$ $\boxed{r^k \cdot r^q = r^{k+q}}$ Lemma 0 — pf by induction.

By induction (assuming Lemma) $(r^n)^m = r^{n \times m}$.

(b) Assume $r \neq 0$, $p, q \in \mathbb{Z}$ Then $(r^p)^q = r^{p \times q}$ $p = -n$ $n \in \mathbb{N}$
where by definition if $p < 0$, LEMMA 1 $(r^{-1})^n$

Then $r^p = r^{-n} := (r^n)^{-1}$

Case 1: $p, q \geq 0$ was part (a) $\Rightarrow p, q \in \mathbb{N}$.

Case 2 $p, q < 0 \Rightarrow p = -n, q = -m \Rightarrow r^{p \times q} = r^{n \times m} = (r^n)^m$

$(r^p)^q = (r^{-n})^{-m} \underset{def}{=} ((r^{-n})^m)^{-1} \underset{def}{=} (((r^n)^{-1})^m)^{-1} \underset{\text{LEMMA 1}}{=} ((r^n)^m)^{-1})^{-1}$

LEMMA 2 $\boxed{(S^{-1})^{-1} = S}$ $= (r^n)^m = r^{n \times m} = r^{p \times q}$

Case 3 $p \geq 0$ $q < 0$ $p = n$, $q = -m \Rightarrow r^{p \times q} = r^{-(n \times m)} = (r^{n \times m})^{-1}$

$(r^p)^q = (r^n)^{-m} = ((r^n)^m)^{-1} = (r^{n \times m})^{-1} = r^{-(n \times m)} = r^{p \times q}$.

Case 4: $p < 0$ $q \geq 0$ $p = -n$, $q = m \Rightarrow r^{p \times q} = r^{-(n \times m)} = (r^{-1})^{n \times m}$

$(r^p)^q = (r^{-n})^q = ((r^{-1})^n)^m = (r^{-1})^{n \times m} = r^{-(n \times m)} = r^{p \times q}$

Lemma 0: $r^k r^l = r^{k+l}$   $P(k)$ induction on $k$

Pf: base case $k=0$   $r^0 \cdot r^l = 1 \cdot r^l = r^l = r^{0+l}$ ✓

Inductive step: $P(k) \Rightarrow P(k+1)$: $r^{k+1} r^l \overset{?}{=} r^{k+1+l}$

now $r^{k+1} \cdot r^l \underset{r^n}{\overset{\text{by def}}{=}} r^k \cdot r^1 \cdot r^l \underset{\downarrow}{=} r^k \cdot r^l \cdot r = r^{k+l} \cdot r$ (IH)

commutativity of multiplication

by def of $r^n$ $= r^{k+l+1} = r^{k+1+l}$ (by comm. of addition in exponents)

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Lemma 1 $(r^n)^{-1} = (r^{-1})^n$   $P(n)$ $r \neq 0$ $n \in \mathbb{N}$

Pf: induction on $n$

base case $n=0$ $P(0)$: $r^0 = 1$, $(r^{-1})^0 = 1$ by def

$\therefore$ $(r^0)^{-1} = 1^{-1} = 1 = (r^{-1})^0$ ✓

Inductive step $P(n) \Rightarrow P(n+1)$: $(r^{n+1})^{-1} \overset{?}{=} (r^{-1})^{n+1}$

$(r^{n+1}) = r^n \cdot r$

$(r^{-1})^{n+1} = (r^{-1})^n \cdot r^{-1} \overset{(IH)}{=} (r^n)^{-1} \cdot r^{-1} = (r^n \cdot r)^{-1} = (r^{n+1})^{-1}$ ✓

Lemma 3 ✓

$s^{-1} \cdot t^{-1} = (s \cdot t)^{-1}$

Lemma 2 $(s^{-1})^{-1} = s$   $s \neq 0$

Pf: $s = a/b$, $a,b \in \mathbb{Z}$ $a,b \neq 0$

$s^{-1} = b/a$ by def

$(s^{-1})^{-1} = a/b = s$ by def ∎

Exercise is now proved and all auxiliary lemmas.

Pf: $s, t \neq 0$

$s = a/b$ $t = c/d$

$a,b,c,d \in \mathbb{Z}$ non-zero

$s^{-1} = b/a$ $t^{-1} = d/c$

$st = \frac{ac}{bd}$ $(st)^{-1} = \frac{bd}{ac}$

$s^{-1} t^{-1} = \frac{b}{a} \cdot \frac{d}{c} = \frac{bd}{ac} = (st)^{-1}$ ✓

∎

(7) $\varepsilon > 0$ $\varepsilon \in \mathbb{Q}$, $x \in \mathbb{Q}$ ~~then~~ show
$\exists n \in \mathbb{N}$ st $|x| < n\varepsilon \iff -n\varepsilon < x < n\varepsilon$

Pf ~~then~~ consider $y = \frac{x}{\varepsilon} \in \mathbb{Q}$ $\varepsilon > 0$.
by interspersing of integers by rationals;
There is and $m \in \mathbb{Z}$ st $m \leq y < m+1$
if $y < 0$ then let $n = |m| + 1$ } $n = |m|+1$ in either case
if $y \geq 0$ then let $n = m+1 > 0$
In either case $|y| < n$
$\Rightarrow \frac{|x|}{\varepsilon} < n \Rightarrow |x| < n\varepsilon$. ☑

(10) Reverse $\Delta$-inequality: $x, y \in \mathbb{Q}$:
$$\big|\,|x| - |y|\,\big| \leq |x-y|$$
Pf: $x = x - y + y \Rightarrow$ by $\Delta$-inequality
$|x| \leq |x-y| + |y| \Rightarrow |x| - |y| \leq |x-y|$

Likewise: $y = y - x + x \Rightarrow$
$|y| \leq |y-x| + |x| \Rightarrow |y| - |x| \leq |y-x|$
$\Rightarrow -(|x| - |y|) \leq |x-y|$

$\therefore -|x-y| \leq |x| - |y| \leq |x-y|$
by property (c) in Prop. 4.3.3
$\therefore \big|\,|x| - |y|\,\big| \leq |x-y|$ ☑

(8) Show $|-x| = |x|$ $\quad x \in \mathbb{Q}$

~~$x \in \mathbb{Q} \Rightarrow x = a/b$ $a, b \in \mathbb{Z}$ $b \neq 0$, $-x = (-a)/b = a/(-b)$~~

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0 \end{cases}$$

$$|-x| = \begin{cases} -x & \text{if } -x \geq 0 \\ -(-x) & \text{if } -x \leq 0 \end{cases} = \begin{cases} -x & \text{if } x \leq 0 \\ x & \text{if } x \geq 0 \end{cases} = |x|$$

(9) (a) $z, x, y \in \mathbb{Q}$ and $|x-y| < 1/2 \Rightarrow |xz - yz| = |z(x-y)| = |z||x-y|$

$\therefore |xz - yz| < |z| \, 1/2 = |z|/2.$

(b) $w, x, y, z \in \mathbb{Q}$ $\quad |w-x| \leq \frac{1}{2}$ $\quad |y-z| \leq \frac{1}{2}$

Then $|(w+y)-(x+z)| = |(w-x)+(y-z)| \underset{\Delta\text{-neq}}{\leq} |w-x| + |y-z|$

$$\leq \frac{1}{2} + \frac{1}{2} = 1.$$

Eg: $x = z = 0$, $w = y = \frac{1}{2}$ Then $|w-x| = \frac{1}{2}$ $\quad |y-z| = \frac{1}{2}$

and $|(w+y)-(x+z)| = |1-0| = 1$