TAKE HOME EXAM # 2 - MATH 401/501 - SPRING 2009

Instructor: Cristina Pereyra

There are six problems worth a total of 60 points, and 4 possible bonus points.. Read carefully all the problems. Choose 4 of the first 5 problems to return with the last problem on Tuesday April 21, 2009, this will be your midterm (50 + 4 points). The other problem will be a bonus homework problem for this week and is due on Thursday April 23, 2009. Please if possible do the last problem on the given page, name all pages. The exam is suppose to show your knowledge and your work, please refrain from copying other people's work. Good luck!!!!

1. (10 points) Let A be a subset of the real numbers bounded above, and let c be a real number. Define the set
$$cA := \{x \in \mathbb{R} : \text{ there exists } a \in A \text{ such that } x = ca\}.$$
Show that if $c \geq 0$ then
$$\sup(cA) = c\sup(A).$$
What can be said when $c < 0$?

2. (10 points) Suppose $\{a_n\}_{n=1}^{\infty}$ is a bounded sequence of real numbers (not necessarily convergent). Assume that $\lim_{n\to\infty} b_n = 0$, show that
$$\lim_{n\to\infty} a_n b_n = 0.$$
If the sequence $\{b_n\}$ is convergent but not to zero, is it still true that the product sequence $\{a_n b_n\}_{n=1}^{\infty}$ converges? If your answer is yes, prove it, and if it is no, give an example.

3. (10 points) Show that if $0 < a_n \leq b_n$ for all n, then

   (a) if $\sum_{n=1}^{\infty} b_n$ converges so does $\sum_{n=1}^{\infty} a_n$.

   (b) if $\sum_{n=1}^{\infty} a_n$ diverges so does $\sum_{n=1}^{\infty} b_n$.

4. (10 points) Is the series
$$\sum_{n=1}^{\infty}\left(\frac{n}{3^n} + \frac{\sqrt{n+1}}{n^{3/2}}\right)$$
convergent? Justify your answer.

5. (10 points) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers convergent to $L \in \mathbb{R}$. Show that the sequence of its averages,
$$b_n = \frac{1}{n}\sum_{k=1}^{n} a_k = \frac{a_1 + a_2 + \cdots + a_n}{n}$$
converges to L as well.

   Is the converse true? Justify your answer, if yes with a proof, if no with an example.

6. (20 points + 4 bonus points) Decide whether the following statements are true or false. Justify your answers with a couple sentences, an example or a reference. (I will count the best 5 towards your 10 points, and if you get all six questions right you get 2 bonus points)

   (a) A finite non-empty set of real numbers always contains its supremum and its infimum.

   ☐ True ☐ False

   (b) Let $\alpha = \limsup_{n\to\infty} a_n$. Then α can never be a lower bound for the sequence $\{a_n\}$.

   ☐ True ☐ False

   (c) There is a bounded sequence that contains no convergent subsequences.

   ☐ True ☐ False

   (d) There is a monotone sequence that diverges but has a convergent subsequence.

   ☐ True ☐ False

   (e) If a series diverges then it must be that the terms are bounded away from zero.

   ☐ True ☐ False

   (f) There are divergent series $\sum x_n$ and $\sum y_n$ such that $\sum x_n y_n$ is convergent.

   ☐ True ☐ False