EXAM # 1 - MATH 401/501 - Spring 2009
February 26, 2009

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The first seven problems are worth a total of 70 points. The last is a bonus problem worth 10 points. Read carefully all the problems. Please do the problems in separate pages, name all the pages and enumerate them. You can use your book and/or notes. Do as much as you can and turn in your work at the end of the exam. Keep the problems and work on them over the weekend (specially those where you think you could have done better than you did in the exam), turn in on Tuesday March 3, 2009 the result of your efforts. Good luck!!!!

1. (10 points) Show by induction that the statement
$$P(n): \quad 1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}$$
is true for all natural numbers n, and any fixed rational number $x \neq 1$.

2. (10 points) Given sets A, B, show that
$$A \cap (A \cup B) = A.$$

3. (10 points) Remember that the set A has the same cardinality as the set B if there exists a bijection $f : A \to B$. Show that if A, B and C are arbitrary sets (finite or not), then,
   (a) A has the same cardinality as A.
   (b) If A has the same cardinality as B, then B has the same cardinality as A.
   (c) If A has the same cardinality as B, and B has the same cardinality as C, then A has the same cardinality as C.

   You can use known facts about bijective functions, make sure you state those facts properly.

4. (10 points) Given functions $f : X \to Y$, $g : Y \to Z$, show that if $g \circ f$ is injective (one-to-one) then f must be injective. Is it true that g must also be injective?

5. (10 points) Denote the cardinality of a finite set C by $\#\,(C)$. Let A and B be finite sets, show that $A \cup B$ and $A \cap B$ are finite sets, and that
$$\#\,(A) + \#\,(B) = \#\,(A \cup B) + \#\,(A \cap B).$$

6. (10 points) Given a rational number x, and natural numbers n and m, remember that $x^0 := 1$, and if x^n is defined to be a rational number, then $x^{n+1} := x^n x$, this recursive definition guarantees that if $x \in \mathbb{Q}$ then $x^n \in \mathbb{Q}$ for all $n \in \mathbb{N}$. Show that
$$x^{n+m} = x^n x^m.$$
Hint: fix one of the natural numbers and induct on the other. (Here we are assuming, according to Tao's definition that $0^0 := 1$).

7. (10 points) In this problem, use any property of absolute value you wish, make sure you state properly the properties you are using.

   (a) Show that if $x, y, z \in \mathbb{Q}$, $|x - y| < 1/2$, and $0 < z$ then
   $$|xz - yz| < |z|/2.$$

   (b) Show that if $w, x, y, z \in \mathbb{Q}$, $|w - x| \leq 1/2$ and $|y - z| \leq 1/2$ then
   $$|(w + y) - (x + z)| \leq 1.$$

   Can you find rational numbers x, w, x, y, z such that the hypothesis are satisfied and $|(w + y) - (x + z)| = 1$?

8. (Bonus 10pts) Show that given a positive rational number x, there exists a natural number $n > 0$ such that
$$0 < \frac{1}{n} < x.$$