Real Analysis Quiz 2 Solutions (or outlines at least)

1. Let $a \in \mathbb{R}$ be a positive real number. Let $\{x_n\}$ be the sequence defined by

$$x_0 = 0, \quad x_{n+1} = a + x_n^2, \quad n \geq 0.$$

If the sequence converges what will it converge to? If you are feeling slightly more ambitious find all possible values of a for which the sequence converges and prove your claim.

Solution: If the limit exists, call it x_∞, we must have $x_\infty = a + x_\infty^2$. So using the quadratic equation we get

$$x_\infty = \frac{1 \pm \sqrt{1 - 4a}}{2}.$$

Therefore, $a \leq 1/4$. For the convergence part assume $0 < a \leq 1/4$, by hypothesis. The sequence is non-decreasing. Using the value for x_∞ we see that

$$x_{n+1} = a + x_n^2 < \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

Therefore, we see that the sequence is bounded. So we have a bounded sequence that is non-decreasing. Hence is must converge.

2. Let $\{a_n\}$ and $\{b_n\}$ be two Cauchy sequences of real numbers. Then show that the sequence $\{|a_n - b_n|\}$ is Cauchy.

Solution: Since $\{a_n\}$ and $\{b_n\}$ are both Cauchy sequences there exists an N such that for all $n > N$ we have

$$|a_n - a_{n+1}| < \frac{\epsilon}{2}$$

and

$$|b_n - b_{n+1}| < \frac{\epsilon}{2}.$$

Now it follows that

$$|a_{n+1} - b_{n+1}| + |a_n - a_{n+1}| + |b_n - b_{n+1}| > |a_n - b_n|$$

and also

$$|a_n - b_n| + |a_n - a_{n+1}| + |b_n - b_{n+1}| > |a_{n+1} - b_{n+1}|$$

by the triangle inequality. So we get that

$$\epsilon > |a_n - a_{n+1}| + |b_n - b_{n+1}| > \Big||a_{n+1} - b_{n+1}| - |a_n, b_n|\Big|,$$

That is, $\{|a_n - b_n|\}$ is a Cauchy sequence as claimed.

3. (6.6.2) Find two sequences such that each is a subsequence of the other, but they are not equal.

Solution: The sequences $\{1, 2, 1, 2, 1, 2, \dots\}$ and $\{2, 1, 2, 1, 2, 1, \dots\}$ work. If you know of a more interesting example please tell me.

4. (7.3.2) Let x be a real number with $|x| < 1$. Then show that

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1 - x}.$$

What can you say when $|x| \geq 1$?

Solution: Let $S_n = 1 + x + \cdots + x^n$. Then $(1 - x)S_n = S_n - xS_n = 1 - x^{n+1}$. Therefore,

$$S_n = \frac{1 - x^{n+1}}{1 - x}.$$

Letting $n \to \infty$ we see that $x^{n+1} \to 0$ as $|x| < 1$. Which proves our claim. For $|x| \geq 1$, the series diverges.

5. (7.2.1) Does the series

$$\sum_{n=0}^{\infty}(-1)^n$$

converge or diverge? If it converges can you find its value?

Solution: The series does not converge as the sequene $\{(-1)^n\}$ does not go to zero as $n \to \infty$, by corollary 7.2.6. For the second part use the ratio test to conclude that the series converges for $|x| < 1/3$. To deal with the end-points, apply the p-series test. I got the series converges for $x \in (-1/3, 1/3]$.

6. Construct a sequence $\{x_n\}$ of real numbers satisfying:

$$\limsup_n x_n = 2006 \quad \text{and} \quad \liminf_n x_n = -\infty.$$

Solution: The sequence $\{2006, -1, 2006, -2, 2006, -3, \ldots, 2006, -n, \ldots\}$ works.

7. Let $\{x_n\}$ be a bounded sequence of real numbers. Show that the sequence $\{y_n\}$ given by

$$y_n = \sup_{m \geq n} x_m$$

converges.

Solution: Since the sequence $\{x_n\}$ is bounded we know the sequence $\{y_n\}$ is. Also the sequence $\{y_n\}$ is monotonically decreasing as we have less terms to take the supremum of each time we increase n. Therefore, it converges.

8. Let $x \in \mathbb{Q}$ with $x^2 \leq 2$. Show that we can find a $y \in \mathbb{Q}$ with $x < y$ and $y^2 \leq 2$. Hence, the set $\{x \in \mathbb{Q} : x^2 < 2\}$ has no supremum in $\mathbb{Q}$. Hint: write $x = a/b$ and compare

$$\frac{a}{b} \quad \text{with} \quad \frac{3a + 4b}{2a + 3b}.$$

This comparison was known to Euclid.

Solution: A quick computation shows that if $x = a/b$ and $x^2 \leq 2$, then taking y to be $\frac{3a+4b}{2a+3b}$ gives us the y we are looking for in the problem. That is,

$$\frac{a}{b} < \frac{3a + 4b}{2a + 3b}$$