Real Analysis Quiz 2

1. Let $a \in \mathbb{R}$ be a positive real number. Let $\{x_n\}$ be the sequence defined by

$$x_0 = 0, \quad x_{n+1} = a + x_n^2, \quad n \geq 0.$$

If the sequence converges what will it converge to? If you are feeling slightly more ambitious find all possible values of a for which the sequence converges and prove you claim.

2. Let $\{a_n\}$ and $\{b_n\}$ be two Cauchy sequences of real numbers. Then show that the sequence $\{|a_n - b_n|\}$ is Cauchy.

3. (6.6.2) Find two sequences such that each is a subsequence of the other, but they are not equal.

4. (7.3.2) Let x be a real number with $|x| < 1$. Then show that

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

What can you say when $|x| \geq 1$?

5. (7.2.1) Do the series

$$\sum_{n=0}^{\infty} (-1)^n \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{\sqrt{n}} x^n$$

converge or diverge? For the second one what can you say about its radius of convergence?

6. Construct a sequence $\{x_n\}$ of real numbers satisfying:

$$\limsup_n x_n = 2006 \quad \text{and} \quad \liminf_n x_n = -\infty.$$

7. Let $\{x_n\}$ be a bounded sequence of real numbers. Show that the sequence $\{y_n\}$ given by

$$y_n = \sup_{m \geq n} x_m$$

converges.

8. Let $x \in \mathbb{Q}$ with $x^2 \leq 2$. Show that we can find a $y \in \mathbb{Q}$ with $x < y$ and $y^2 \leq 2$. Hence, the set $\{x \in \mathbb{Q} : x^2 < 2\}$ has no supremum in $\mathbb{Q}$. Hint: write $x = a/b$ and compare

$$\frac{a}{b} \quad \text{with} \quad \frac{3a + 4b}{2a + 3b}.$$

This comparison was known to Euclid.