FINAL EXAM - MATH 401/501 - Fall 2016

December 13, 2016

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There are 8 problems for a total of 110 points. Justifications and well thought arguments are expected. You can use your textbook, no notebooks or calculators are allowed. Good Luck!!

1. (15 points) Given a statement do as indicated in each case.

   (a) Given the statement, where here $f : \mathbb{R} \to \mathbb{R}$:

   $\forall \epsilon > 0\ \exists \delta > 0$ *such that* $\forall x \in \mathbb{R}$ *with* $|x - x_0| \le \delta$ *we have* $|f(x) - L| \le \epsilon$.

   Write the **negation** of the given statement:

   ____

   The **given statement** is the definition of: ________________________

   (b) Write the **contrapositive** of the following statement:

   If the series $\sum_{n=1}^{\infty} a_n$ *converges then* $\lim_{n\to\infty} a_n = 0$.

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   Is the **contrapositive** a true or a false statement? TRUE ☐ FALSE ☐

   Is the **original statement** a true or a false statement? TRUE ☐ FALSE ☐

   (c) Write the **converse** of the following statement:

   If a function $f : \mathbb{R} \to \mathbb{R}$ *is differentiable then it is continuous.*

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   Is the **converse** a true or a false statement? TRUE ☐ FALSE ☐

   Is the **original statement** a true or a false statement? TRUE ☐ FALSE ☐

2. (10 points) Let $a, b \in \mathbb{R}$, and let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = ax + b$.

   (a) Use the ϵ-δ definition of continuity to show that f is continuous at a point $x_0 \in \mathbb{R}$.

   (b) Use the definition of derivative to calculate $f'(x_0)$ at a point $x_0 \in \mathbb{R}$.

3. (10 points) Let $f : [-1, 1] \to \mathbb{R}$ be given by $f(x) = \begin{cases} 1 & \text{if } -1 \le x \le 0 \\ (3x+1)^2 & \text{if } 0 < x \le 1 \end{cases}$

   (a) Is f is differentiable on $[-1, 1]$? Explain. YES ☐ NO ☐

   (b) Find $f'(x_0)$ for each point $x_0 \in [-1, 1]$ at which f is differentiable.

4. (10 points) Let $f, g : \mathbb{R} \to \mathbb{R}.$ and let $x_0 \in \mathbb{R}.$ Assume that $\lim_{x \to x_0} g(x) = 0.$

   (a) Show that if $\lim_{x \to x_0} f(x)$ exists then $\lim_{x \to x_0} g(x)\, f(x) = 0.$

   (b) Show that if $\lim_{x \to x_0} f(x)$ does not exist but f is a bounded function on $\mathbb{R}$ then

$$\lim_{x \to x_0} g(x)\, f(x) = 0.$$

5. (10 points) Assume $f, g : \mathbb{R} \to \mathbb{R}$ are uniformly continuous on $\mathbb{R}$.

   (a) Use the ϵ-δ definition of uniform continuity to show that $f + g$ is uniformly continuous on $\mathbb{R}$.

   (b) Show that the composition $(g \circ f)$ is uniformly continuous on $\mathbb{R}$.

6. (30 points) Decide whether the following statements are true or false. All the functions and sequences considered are defined on $\mathbb{R}$, unless stated differently. Justify your answers with a couple sentences, a counterexample, or a reference. If the statement is false, modify it to make it true.

   (a) If a series of real numbers $\sum_{k=0}^{\infty} a_k$ converges then $\sum_{k=0}^{\infty} |a_k|$ also converges.

   ☐ True ☐ False

   (b) Continuous functions attain their maximum and minimum values on their domain.

   ☐ True ☐ False

(c) Given an increasing sequence $\{a_n\}_{n\geq 1}$ bounded above, and given a continuous decreasing function $f : \mathbb{R} \to \mathbb{R}$, then $f(\sup\{a_n\}_{n\geq 1}) = \inf\{f(a_n)\}_{n\geq 1}$.

☐ True ☐ False

(d) If f is a differentiable and strictly decreasing function then $f'(x) < 0$ for all $x \in \mathbb{R}$.

☐ True ☐ False

(e) There is some $x \in [0, 1]$ such that $f(x) = 2^x - 3x = 0$.

☐ True ☐ False

(f) $\displaystyle\int_0^1 \frac{x}{(x^2+1)^2} dx = \frac{1}{4}$ ☐ True ☐ False

7. (15 points) Assume the function $f : \mathbb{R} \to \mathbb{R}$ satisfies the following additive property

$$f(x + y) = f(x) + f(y) \quad \text{for all } x, y \in \mathbb{R}.$$

(a) Show that $f(0) = 0$ and that $f(-x) = -f(x)$ for all $x \in \mathbb{R}$.

(b) Let $a := f(1)$. Show (by induction) that $f(n) = an$ for all natural numbers $n \in \mathbb{N}$.

(c) Show that $f(k) = ak$ for all integer numbers $k \in \mathbb{Z}$, where $a = f(1)$.

(d) Show that $f(1/m) = a/m$ for all $m \neq 0$ and $m \in \mathbb{N}$, where $a = f(1)$. Now show that $f(r) = ar$ for all rational numbers $r \in \mathbb{Q}$.

(e) Show[1] that if f is continuous on $\mathbb{R}$ then $f(x) = ax$ for all real numbers $x \in \mathbb{R}$.

[1]For each part of this and the next problem you can assume the previous parts are true.

8. (10 points) Given function $f : [a, b] \to [a, b]$.

   The function f is a *contraction* on $[a, b]$ if there is $0 < \alpha < 1$ such that

$$|f(x) - f(y)| \leq \alpha |x - y| \quad \text{for all} \;\; x, y \in [a, b].$$

   (a) A point $x \in [a, b]$ is a *fixed point* for f if and only if $f(x) = x$. Show that if f is a contraction on $[a, b]$ then there is at most one fixed point for f. [Hint: Assume there are two different fixed points $x, y \in [a, b]$ and reach a contradiction.]

   (b) If f is a contraction on $[a, b]$, pick any point $x_0 \in [a, b]$, let $x_1 := f(x_0) \in [a, b]$, and given $x_n \in [a, b]$ define recursively $x_{n+1} := f(x_n) \in [a, b]$. Show that the sequence $\{x_n\}_{n \geq 0}$ in $[a, b]$ is convergent to a point $x \in [a, b]$. [Hint: show that the sequence is a Cauchy sequence.]

   (c) Let $\{x_n\}_{n \geq 0}$ be the convergent sequence defined in (b). Show that $x = \lim_{n \to \infty} x_n$ is a *fixed point* for f, that is, show that $f(x) = x$.

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1			2		3		4		5		6						7					8			
a.	b.	c.	a.	b.	a.	b.	a.	b.	a.	b.	a.	b.	c.	d.	e.	f.	a.	b.	c.	d.	e.	a.	b.	c.	Total