SOLUTIONS - Practice Problems Midterm 2 - Math 401/501 Nov 4, 2014 ①

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1. (a) $p \in \mathbb{Q}$, $p \neq 0$, $x \in \mathbb{R} \setminus \mathbb{Q}$ (x is irrational) show $px \in \mathbb{R} \setminus \mathbb{Q}$.

Contradiction: suppose $px = q \in \mathbb{Q}$ then $x = q/p \in \mathbb{Q} \rightarrow\leftarrow$

$\therefore$ px must be irrational. //

(b) if $x, y \in \mathbb{R}$ $x < y$ find irrational $w : x < w < y$ (irrationals are dense in $\mathbb{R}$).

$\frac{x}{\sqrt{2}}, \frac{y}{\sqrt{2}} \in \mathbb{R}$ and $\frac{x}{\sqrt{2}} < \frac{y}{\sqrt{2}}$, since rationals are dense in $\mathbb{R}$, $\exists p \in \mathbb{Q}$ st $\frac{x}{\sqrt{2}} < p < \frac{y}{\sqrt{2}}$

$\therefore$ $x < p\sqrt{2} < y$, let $w \doteq p\sqrt{2}$, by part (a) since $p \in \mathbb{Q}$, $\sqrt{2} \notin \mathbb{Q}$ then $w \notin \mathbb{Q}$. //

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2. (a) $A = \{1, -\frac{1}{2}, 3\}$. since $-\frac{1}{2} \leq -\frac{1}{2} < 1 < 3 \leq 3$ then

A is bounded below by $-\frac{1}{2} \in A$, so $\boxed{\inf A = -\frac{1}{2}}$.

is bounded above by $3 \in A$, so $\boxed{\sup A = 3}$.

(b) $A = \{\frac{n}{n+1} : n \in \mathbb{N}, n \geq 1\} = \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots\}$

The sequence $a_n = \frac{n}{n+1}$ is increasing, since

$a_n = \frac{n}{n+1} = 1 - \frac{1}{n+1}$, $n < n+1$ so $\frac{1}{n+1} < \frac{1}{n}$ and $-\frac{1}{n} < -\frac{1}{n+1}$

finally $a_{n-1} = 1 - \frac{1}{n} < 1 - \frac{1}{n+1} = a_n$ $\forall n \geq 2$.

This means A is bounded below by $a_1 = \frac{1}{2} \in A$

so $\boxed{\inf A = \frac{1}{2}.}$

On the other hand since $a_n = 1 - \frac{1}{n+1} < 1$ $\forall n \geq 1$,

then A is bounded above by 1.

2(b) (continuation) <u>Claim</u> $\sup A = 1$. We need to show that 1 is the least upper bound. Suppose is not, then there is $1 - \varepsilon < 1$ that is an upper bound of A for some $\varepsilon > 0$. But by Archimedean principle $\exists N > 0$, $N \in \mathbb{N}$ st $0 < \frac{1}{N+1} < \varepsilon$ and then $1 - \varepsilon < 1 - \frac{1}{N+1} = a_N < 1$, so $a_N \le 1-\varepsilon < a_N$ $\rightarrow\leftarrow$ contradiction. Thus 1 must be the l.u.b of A, and $\boxed{\sup A = 1}$. //

2(c) $A = \{r \in \mathbb{Q} : r < 5\}$

by definition if $r \in A$ then $r < 5$, so A is bounded above by 5. Given any $M < 5$, M cannot be an upper bound of A because by density of $\mathbb{Q}$ in $\mathbb{R}$, there is $r \in \mathbb{Q}$ such that $M < r < 5$. Thus $\boxed{\sup A = 5}$

A is not bounded below, since given any $M < 5$, $\exists N \in \mathbb{Z} \subset \mathbb{Q}$ such that $N < M < 5$, so $N \in A$ and M cannot be a lower bound for A. So $\boxed{\inf A = -\infty}$

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3. $E \neq \emptyset$ E is bounded subset of $\mathbb{R}$, so $\exists M \in \mathbb{R}$ $\forall x \in E$, $|x| \le M$. Define $\lambda E = \{\lambda x : x \in E\}$ for $\lambda < 0$. For all $x \in E$ $x \le \sup(E)$, $\lambda < 0$ then $\lambda x \geq \lambda \sup(E)$ so $\lambda \sup(E)$ is a lower bound for λE. Since $\inf(\lambda E)$ is the greatest lower bound for λE then $\lambda \sup(E) \le \inf(\lambda E)$.

3. (continuation)

Suppose $\lambda \sup(E) < \inf(\lambda E) \leq \lambda x \quad \forall x \in E$

since $\lambda < 0$, $\frac{1}{\lambda} < 0$ so

$$\sup(E) > \frac{1}{\lambda} \inf(\lambda E) \geq x \quad \forall x \in E$$

So $\frac{1}{\lambda} \inf(\lambda E)$ is an upper bound for E that is smaller than the l.u.b. $\sup(E)$, that is a contradiction. It must be that

$$\lambda \sup(E) = \inf(\lambda E).$$

Using a mirror argument (homework problem $\sup(-E) = -\inf(E)$, $\inf(-E) = -\sup(E)$) we conclude

$$\boxed{\sup(\lambda E) = \lambda \inf(E)}$$

4. If $A, B \neq \emptyset$, bounded subsets of $\mathbb{R}$ with $A \subseteq B$, then $\forall x \in B$ $x \geq \inf B$, in particular $\forall x \in A \subseteq B$, $x \geq \inf B$, so $\inf B$ is a ~~upper~~ lower bound for A, but $\inf A$ is the greatest lower bound of A so $\boxed{\inf B \leq \inf A}$

5. (a) TRUE, $\forall \varepsilon > 0\ \exists N > 0$ s.t. $\forall n \geq N$ $|x_n - x| \leq \varepsilon$, Given $\varepsilon > 0$ for the same $N > 0$ $\forall n \geq N$, by reverse Δ-inequality,

$$\big||x_n| - |x|\big| \leq |x_n - x| \leq \varepsilon,$$

so $\lim_{n\to\infty} |x_n| = |x|$ by definition

(b) FALSE, consider $x_n = (-1)^n$, then $|x_n| = 1$ and $\lim_{n\to\infty} |x_n| = 1$, however $\lim_{n\to\infty} (-1)^n$ doesn't exist.

6. (a) $\lim_{n\to\infty} y_n = -\infty$ iff for all $M<0$ there is $N>0$ s.t. for all $n \geq N$ $x_n \leq M$. //

(b) Assume $x_n \leq z_n$ $\forall n \geq 0$ and $\lim_{n\to\infty} x_n = +\infty$ then $\forall M>0$ $\exists N>0$ s.t. for all $n \geq N$

$$M \leq x_n \leq z_n \;, \Rightarrow \lim_{n\to\infty} z_n = +\infty \;. //$$

(c) $x_n \geq 0$ ($\Rightarrow$) If $\lim_{n\to\infty} x_n = +\infty$ then $\forall M>0$ $\exists N>0$ st $\forall n \geq N$, $M \leq x_n$,

$$\Rightarrow 0 < \frac{1}{x_n} \leq \frac{1}{M} \quad \forall n \geq N$$

Given $\varepsilon>0$ $\exists M>0$ st $0 < \frac{1}{M} < \varepsilon$ (Archimedean Principle). (or just take $M = 1/\varepsilon$)

$\exists N \geq 0$ st $\forall n \geq N$

$$0 < \frac{1}{x_n} \leq \frac{1}{M} < \varepsilon \;, \text{ by def. } \lim_{n\to\infty} \frac{1}{x_n} = 0.$$

($\Leftarrow$) If $\lim_{n\to\infty} \frac{1}{x_n} = 0$, $\forall \varepsilon>0$ $\exists N>0$ st

$$0 < \frac{1}{x_n} < \varepsilon \quad \forall n \geq N.$$

Given $M>0$, let $\varepsilon = 1/M > 0$ Then $\forall n \geq N$

$$\frac{1}{x_n} \leq \frac{1}{M} \Rightarrow M \leq x_n \quad \forall n \geq N$$

so by definition $\lim_{n\to\infty} x_n = +\infty$. //

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7. $t_n > 0$ $\quad \lim_{n\to\infty} t_n = t \geq 0$

(a) Convergent, moreover $\boxed{\lim_{n\to\infty} \sqrt{t_n} = \sqrt{t}}$

<u>Case 1</u>: $t=0$, $\lim_{n\to\infty} t_n = 0$ means $\forall \varepsilon_0>0$ $\exists N>0$ st $\forall n \geq N$ $0 < t_n \leq \varepsilon_0$, then $0 < \sqrt{t_n} \leq \sqrt{\varepsilon_0}$, given $\varepsilon>0$ let $\varepsilon_0 = \varepsilon^2 > 0$, then $\forall n \geq N$ $\sqrt{t_n} \leq \sqrt{\varepsilon^2} = \varepsilon \Rightarrow \lim_{n\to\infty} \sqrt{t_n} = 0$.

7.(a) (continuation)

Case 2: $t>0$, $\lim_{n\to} t_n = t$ means $\forall \varepsilon_0 > 0\ \exists N>0$ st $\forall n \geq N$ $|t_n - t| \leq \varepsilon_0$.

Given $\varepsilon > 0$ let $\varepsilon_0 = \varepsilon\sqrt{t} > 0$ then $\forall n \geq N$

$$|\sqrt{t_n} - \sqrt{t}| = \frac{|t_n - t|}{\sqrt{t_n}+\sqrt{t}} \leq \frac{|t_n - t|}{\sqrt{t}} \leq \frac{\varepsilon_0}{\sqrt{t}} = \frac{\varepsilon\cancel{\sqrt{t}}}{\cancel{\sqrt{t}}} = \varepsilon$$

by definition $\lim_{n\to\infty}\sqrt{t_n} = \sqrt{t}$ //

(b) $\boxed{\lim_{n\to\infty} b_n} = \lim_{n\to\infty}(5t_n^3 - t_n^2 + 7) = \boxed{5t^3 - t^2 + t}$

by limit laws

$\lim_{n\to\infty} t_n^k = t^k$

(c) $\boxed{\lim_{n\to\infty} c_n = 0}$ by squeeze Theorem

Remember from 1st homework $n^2 \leq 2^n\ \forall n \geq 4$

Then $\frac{n}{2^n} \leq \frac{1}{n} \Rightarrow -\frac{1}{n} \leq \frac{n}{2^n}(-1)^n \leq \frac{1}{n}$

$-\frac{1}{n} \xrightarrow{n\to\infty} 0$, $\frac{n}{2^n}(-1)^n = c_n$, $\frac{1}{n} \xrightarrow{n\to\infty} 0$

So by squeeze thm $\lim_{n\to\infty} c_n$ exists and equals 0.

(d) $\boxed{\lim_{n\to\infty} d_n = +\infty}$

Given $M>0$ $\exists N_1 \in \mathbb{N}$ st $N_1 \leq M < N_1 + 1$

since $\{t_n\}$ is convergent, it is bounded $-N_0 \leq t_n \leq N_0 \in \mathbb{N}$

Let $N = N_1 + 1 + N_0$

$\forall n \geq N$ $n + t_n \geq N - N_0 = N_1 + 1 > M$

$\therefore \lim_{n\to\infty}(n+t_n) = +\infty$ //

8. $x_1 := 1$, $x_{n+1} := \sqrt{1+x_n}$ $n \geq 1$

$\left\{ \begin{array}{l} x_1 = 1 \\ x_2 = \sqrt{2} \\ x_3 = \sqrt{1+\sqrt{2}} \\ x_4 = \sqrt{1+\sqrt{1+\sqrt{2}}} \end{array} \right.$

Claim 1: $\{x_n\}$ is strictly increasing.

Pf We will show that $\boxed{x_n < x_{n+1}}$ $P(n)$ by induction on n.

Base case $n=1$: $x_1 = 1 < \sqrt{2} = x_2$ ✓.

Inductive step: assume $x_n < x_{n+1}$ prove $x_{n+1} < x_{n+2}$

$$x_{n+2} = \sqrt{1+x_{n+1}} \underset{IH}{>} \sqrt{1+x_n} = x_{n+1} \checkmark.$$

by induction $\{x_n\}$ is strictly increasing //.

Claim 2: $\{x_n\}$ is bounded by 2.

Pf: by induction on n: $Q(n)$: $x_n < 2$

Base case $n=1$: $x_1 = 1 < 2$ ✓.

Inductive step: assume $x_n < 2$, prove $x_{n+1} < 2$

by def. $x_{n+1} = \sqrt{1+x_n} \underset{I.H}{<} \sqrt{1+2} = \sqrt{3} < 2$.

So $\forall n \geq 1$ $x_n < 2$. Notice $x_n > 1$ $\forall n$

The sequence $\{x_n\}$ is increasing and bounded above, so by the Monotone Bounded Sequence Thm $\{x_n\}$ must converge to some $L \in \mathbb{R}$,

so $\boxed{\lim_{n\to\infty} x_n = L}$, then $\lim_{n\to\infty} x_{n+1} = L \geq 1$

also $L = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \sqrt{1+x_n} = \sqrt{1+L}$ by Problem 7(a)

$\therefore$ $L^2 = 1+L$, $L^2 - L - 1 = 0 \Rightarrow \boxed{L = \frac{1 + \sqrt{5}}{2}} > 1$ //

9. $X_n = n \sin^2(n\pi/2)$

$X_0 = 0$, $X_1 = 1$ $X_2 = 0$ $X_3 = 3(-1)^2 = 3$, etc

Sequence $0, 1, 0, 3, 0, 5, 0, 7, 0, 9, 0, 11, \ldots$

The subsequence of even terms $X_{2k} = 0$ $\forall k$ converges to 0, $\lim_{k\to\infty} X_{2k} = 0$.

The subsequence of odd terms $X_{2k+1} = 2k+1$ $\forall k$ diverges to $+\infty$ $\lim_{k\to\infty} X_{2k+1} = +\infty$.

$\boxed{S = \{0, +\infty\}}$ because no other $c \in \mathbb{R}$ can be a limit point.

If $c \neq 0$, Then if $\underline{c < 0}$ choose $\varepsilon = \frac{|c|}{2} > 0$

Then $c + \varepsilon = c + \frac{|c|}{2} = c - \frac{c}{2} = \frac{c}{2} < 0$

$\{x_n : n \geq 0\} \cap [c-\varepsilon, c+\varepsilon] = \emptyset$

if $\underline{c > 1}$, let $\varepsilon = \frac{1}{4}$ Then $\{x_n : n \geq 0\} \cap [c - \frac{1}{4}, c + \frac{1}{4}]$ contains at most 1 natural positive number which may (if odd) or not (if even) be in the sequence.

In both cases $\# \{x_n : n \geq 0\} \cap [c - \frac{1}{4}, c + \frac{1}{4}] \leq 1$

if $0 < c < 1$ let $\varepsilon = \frac{|c|}{2}$ then $c - \varepsilon = c - \frac{c}{2} = \frac{c}{2} > 0$.

so $\# \{x_n : n \geq 0\} \cap [c - \varepsilon, c + \varepsilon] \leq 1$.

In all cases $c < 0$ or $c > 0$ cannot be a limit point for the sequence.

$\boxed{\limsup \{x_n\} = +\infty, \quad \liminf \{x_n\} = 0}$

10. $S_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ $\quad n \geq 1$

First note that for all $n \geq 1$, $S_{2n} - S_n \geq \frac{1}{2}$, because

$$S_{2n} - S_n = \underbrace{\frac{1}{2n} + \frac{1}{2n-1} + \cdots + \frac{1}{n+1}}_{n \text{ terms}} \geq \underbrace{\frac{1}{2n} + \frac{1}{2n} + \cdots + \frac{1}{2n}}_{n \text{ terms}} = \frac{n}{2n} = \frac{1}{2}$$

Suppose $\{S_n\}$ converges then it must be Cauchy.

For $\varepsilon = \frac{1}{4}$ $\exists N > 0$ $\forall n, m \geq N$ then $|S_m - S_n| \leq \frac{1}{4}$

Let $n \geq N$ and $m = 2n > n \geq N$

$$\frac{1}{2} < S_{2n} - S_n = S_m - S_n \leq \frac{1}{4} \quad \text{so} \quad \cancel{\frac{1}{2} < \frac{1}{4}}$$

contradiction. Hence $\{S_n\}$ is not convergent.

So series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent $\boxtimes$.

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11. $\{S_n\}_{n \geq 0}$ is contractive, so $\exists r$ $0 < r < 1$

s.t. $|S_{n+2} - S_{n+1}| \leq r\,|S_{n+1} - S_n|$ $\forall n \geq 0$.

<u>Lemma</u>: $\{S_n\}$ contractive then $\forall m \geq 0$

$$|S_{m+1} - S_m| \leq r^m |S_1 - S_0| \quad \leftarrow P(m)$$

<u>Pf</u>: Induction on $m \geq 0$.

<u>Base Case:</u> $m = 0$ $\quad |S_1 - S_0| \leq |S_1 - S_0| = r^0 |S_1 - S_0|$.

<u>Inductive step:</u> Assume $P(m)$ prove $P(m+1)$: $|S_{m+2} - S_{m+1}| \leq r^{m+1} |S_1 - S_0|$ ✓

$$|S_{m+2} - S_{m+1}| \underset{\text{def. contractive}}{\leq} r\,|S_{m+1} - S_m| \underset{\text{I.H.}}{\leq} r \cdot r^m |S_1 - S_0|$$

$$= r^{m+1} |S_1 - S_0|$$

done. //

11. (continuation). Want to show $\{s_n\}$ is Cauchy.

suppose $m > n \geq N$ then

$$|s_m - s_n| \underset{\Delta\text{-ineq}}{\leq} |s_m - s_{m-1}| + |s_{m-1} - s_{m-2}| + \dots + |s_{n+1} - s_n|$$

$$\underset{\text{Lemma}}{\leq} (r^{m-1} + r^{m-2} + \dots + r^n)\, |s_1 - s_0|$$

$n \geq N$, $0 < r < 1$, $r^n \leq r^N$

$$\leq r^n (1 + r + r^2 + \dots)\, |s_1 - s_0|$$

geom. series

$$\leq r^N \frac{1}{1-r} |s_1 - s_0|$$

Since $|r| < 1$ $\lim_{N \to \infty} r^N = 0$, so given $\varepsilon_0 > 0$

$\exists N_1 > 0$ st $\forall N \geq N_1$ $r^N \leq \varepsilon_0$

Given $\varepsilon > 0$, let $\varepsilon_0 = \frac{\varepsilon}{|s_1 - s_0|} |1 - r| > 0$

let $N = N_1$ Then $\forall m > n \geq N$

$$|s_m - s_n| \leq \frac{r^N}{1-r} |s_1 - s_0| \leq \frac{\varepsilon_0}{1-r} |s_1 - s_0| = \varepsilon$$

so we have shown $\{s_n\}$ is Cauchy. //

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12. $\lim_{n \to \infty} r_n = r > 0$, $|s_n| \leq M$ $\forall n > 0$.

Show that $\limsup r_n s_n = r \limsup s_n$.

$\exists$ subsequence $\{s_{n_k}\}_{k \geq 0}$ that converges to $\limsup s_n$

12. (continuation) $\lim_{k\to\infty} s_{n_k} = \limsup s_n$

Since $\lim_{n\to\infty} r_n = r$, then all subsequences of $\{r_n\}$ converge to r. In particular $\lim_{k\to\infty} r_{n_k} = r$

$\therefore$ by limit laws

$$\lim_{k\to\infty} r_{n_k} s_{n_k} = \lim_{k\to\infty} r_{n_k} \lim_{k\to\infty} s_{n_k}$$
$$= r \cdot \limsup(s_n)$$

So the subsequence $\{r_{n_k} s_{n_k}\}_{k\geq 0}$ of $\{r_n s_n\}$ converges to $r \cdot \limsup s_n$. Since $\limsup(r_n s_n)$ is the largest subsequential limit it must be that

$$r \cdot \limsup(s_n) \leq \limsup(r_n s_n)$$

Suppose $r \cdot \limsup(s_n) < \limsup(r_n s_n)$

Then there must exist another subsequence $\{r_{n'_k} s_{n'_k}\}$ of $\{r_n s_n\}$ that converges to $\limsup(r_n s_n)$

$$\lim_{k\to\infty} r_{n'_k} s_{n'_k} = \limsup(r_n s_n)$$

since $0 < r = \lim_{k\to\infty} r_{n'_k}$, ~~we can assume that~~ then eventually $r_{n'_k} > 0$

so by limit laws

$$\lim_{k\to\infty} s_{n'_k} = \lim_{k\to\infty} \frac{r_{n'_k} s_{n'_k}}{r_{n'_k}} = \frac{\lim_{k\to\infty} r_{n'_k} s_{n'_k}}{\lim_{k\to\infty} r_{n'_k}} = \frac{\limsup(r_n s_n)}{r}$$

So $\frac{\limsup(r_n s_n)}{r}$ is a subsequential limit for $\{s_n\}$

Since $\limsup(s_n)$ is the largest subsequential limit for $\{s_n\}$ it must be that

$$\frac{\limsup(r_n s_n)}{r} \le \limsup(s_n) < \frac{\limsup(r_n s_n)}{r}$$

$\rightarrow \leftarrow$ contradiction so it must be that $\boxed{\limsup(r_n s_n) = r \limsup(s_n)}$.

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13. $\sum_{n=0}^{\infty} a_n$ convergent iff given $\varepsilon > 0$ $\exists N > 0$ st $\forall q \geq p \geq N$ $\left|\sum_{n=p}^{q} a_n\right| \le \varepsilon$. In particular

if $q = p \geq N$ $\left|\sum_{n=p}^{p} a_n\right| = |a_p| \le \varepsilon$,

by definition $\lim_{p\to\infty} a_p = 0$.

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14. If $0 \le a_n \le b_n$ $\forall n \geq 0$ and $\sum_{n=0}^{\infty} b_n$ converges

Then given $\varepsilon > 0$ $\exists N > 0$ st $\forall q \geq p \geq N$

$\left|\sum_{n=p}^{q} b_n\right| \le \varepsilon$. Note that

$$0 \le \sum_{n=p}^{q} a_n \le \sum_{n=p}^{q} b_n \le \varepsilon \quad \forall p, q \geq N$$

So by Cauchy test $\sum_{n=0}^{\infty} a_n$ is convergent.

15. Assume $\sum_{n=0}^{\infty} |a_n|$ converges $\Rightarrow$ Given $\varepsilon > 0$

$\exists N > 0$ st. for all $q > p \geq N$ $\left| \sum_{n=p}^{q} |a_n| \right| \leq \varepsilon$.

$$\left| \sum_{n=p}^{q} a_n \right| \underset{\Delta\text{-ineq}}{\leq} \sum_{n=p}^{q} |a_n| \leq \varepsilon \quad \forall\, q \geq p \geq N$$

by Cauchy test $\sum_{n=0}^{\infty} a_n$ is convergent.

Lemma (Δ-inequality for ~~m~~ terms) $b_i \in \mathbb{R}$

$P(m)$: $|b_1 + \dots + b_m| \leq |b_1| + |b_2| + \dots + |b_m|$

Pf (Induction on m)

base cases: $m=1$ $|b_1| \leq |b_1|$ done

$m=2$ $|b_1 + b_2| \underset{\Delta\text{-ineq}}{\leq} |b_1| + |b_2|$

Inductive step given $P(m)$ show $P(m+1)$

$$|(b_1 + \dots + b_m) + b_{m+1}| \underset{\Delta\text{-ineq}}{\leq} |b_1 + \dots + b_m| + |b_{m+1}|$$
$$\underset{I.H.}{\leq} (|b_1| + \dots + |b_m|) + |b_{m+1}|$$
$$= |b_1| + \dots + |b_{m+1}|$$

$\boxtimes$