Practice Problems for Exam # 1 - MATH 401/501 - Fall 2012

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The following problems are generally harder than what you will find in a 50 minute exam. However they should provide a good workout and an opportunity for reviewing most of the concepts we have encountered. Part of what you are learning in this class is to be able to work with a new definition, without fear (exercises 4, 5, 10 and 11 are examples of problems where objects you have not encountered before are defined for you and you are asked to prove something about them). In a 50 minute exam I will not use such multistep questions. Exercise 6 is a multistep problem but we did similar things, one part (something like part (a)) could appear in the exam.

1. Show by induction that the statement
$$P(n): \quad 1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}$$
is true for all natural numbers n, and any fixed rational number $x \neq 1$.

2. Given a set X and subsets A and B of X. Show that
$$X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B).$$

3. Let r and q be rational numbers. Show that if $rq = 0$ then $r = 0$ or $q = 0$.

4. Suppose $f : X \to Y$, and suppose that A, B are subsets of X and C, D are subsets of Y. Determine which inclusion relationship must hold for the following pair of sets (are they equal? is one inside the other? which one? Is there a class of functions for which you will get equality in each case? in both cases?):

   (a) $f(A \cap B)$ and $f(A) \cap f(B)$,
   (b) $f^{-1}(C \cap D)$ and $f^{-1}(C) \cap f^{-1}(D)$.

   Remember that for $A \subset X$, $f(A) := \{y \in Y : y = f(x), x \in A\}$ is a subset of Y (the direct image of A under f); and for $C \subset Y$, $f^{-1}(C) := \{x \in X : f(x) \in B\}$ is a subset of X (the inverse image of B under f).

5. Show that given any sets (finite or not) A, B, C such that $B \cap C = \emptyset$ then
$$\#(A^B \times A^C) = \#(A^{B \cup C}).$$
Remember that the set A^B consists of all functions $f : B \to A$.

6. Given a rational number r, and natural numbers n and m. We define $r^0 := 1$ (notice in particular that we are defining $0^0 = 1$) and given the rational number r^n then we define the rational number $r^{n+1} := r^n \times r$.

   (a) Show that
$$(r^n)^m = r^{n \times m}.$$
   Hint: fix one of the natural numbers and induct on the other.

   (b) Assume now that $r \neq 0$, p and q are integers, and show that $(r^p)^q = r^{p \times q}$. Where we define for a negative integer $p = -n$, $n \in \mathbb{N}$, $r^p = r^{-n} := (r^n)^{-1}$ (we are assuming here $r \neq 0$ so that we do not take the reciprocal of zero). Useful auxiliary lemma is to show that: $(r^n)^{-1} = (r^{-1})^n$.

7. Let $\epsilon > 0$ be a positive rational number (a "step" or "unit"). Show that given a positive rational number $x \geq 0$, there exists a natural number n (depending both on the step ϵ and on x) such that $|x| < n\epsilon$. In words: given any positive step size we can overcome any fixed rational number with a finite number of steps.

8. In this problem, use any property of absolute value you wish, make sure you state properly the properties you are using.

   (a) Show that if $x, y, z \in \mathbb{Q}$, $|x - y| < 1/2$, and $0 < z$ then
   $$|xz - yz| < |z|/2.$$

   (b) Show that if $w, x, y, z \in \mathbb{Q}$, $|w - x| \leq 1/2$ and $|y - z| \leq 1/2$ then
   $$|(w + y) - (x + z)| \leq 1.$$

   Can you find rational numbers x, w, x, y, z such that the hypothesis are satisfied and $|(w + y) - (x + z)| = 1$?

9. Show that the "reverse triangle inequality" holds for $x, y \in \mathbb{Q}$:
   $$\big||x| - |y|\big| \leq |x - y|.$$

10. Given $n \geq 1$ rational numbers: $a_1, a_2, a_3, \ldots, a_n$, we define a new rational number A_n, inductively: (i) $A_1 := a_1$, and (ii) Suppose A_n has been defined for $n \geq 1$, then we define
   $$A_{n+1} := \frac{n \times A_n + a_{n+1}}{n + 1}.$$

   (a) Show that A_n is the *arithmetic average* of the n given numbers:
   $$A_n = \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

   (b) Show that every set of n rational numbers $\{a_1, a_2, a_3, \ldots, a_n\}$ contains an element which is larger or equal than their average A_n.

11. Suppose $f : A \to B$ is a bijection and $g : B \to B$. Let h be the composition $h = f^{-1} \circ g \circ f$, so that $h : A \to A$. We define a function $h^{(1)} := h$ (the *first iterate* of h). Suppose we have defined a function $h^{(n)} : A \to A$ (the nth-*iterate* of h), we define a function $h^{(n+1)} : A \to A$ (the $(n + 1)$th-*iterate* of h) by
   $$h^{(n+1)} := h^{(n)} \circ h.$$

   (a) Find $h^{(2)}$ in terms of f and g.

   (b) Derive a formula in terms of f and g for the nth iterate of h (you will have to verify that your formula is correct by induction).