Instructions: Attempt all problems. Only four will be graded. The exercises are in Chapter 1 of our textbook Linear Algebra, by S.H. Friedberg, A.J. Insel, L.E. Spence, Fourth Edition. This homework is due on Tuesday January 26, 2010.

Section 1.2 - Pages 12-15

1. Label the following statements as true or false.
   
(a) Every vector space contains a zero vector.
(b) A vector space may have more than one zero vector.
(c) In any vector space, $ax = bx$ implies that $a = b$.
(d) In any vector space, $ax = ay$ implies that $x = y$.
(e) A vector in $\mathbb{R}^n$ may be regarded as a matrix in $M_{n \times 1}(\mathbb{R})$.
(f) An $m \times n$ matrix has $m$ columns and $n$ rows.
(g) In $P(\mathbb{R})$, only polynomials of the same degree may be added.
(h) If $f$ and $g$ are polynomials of degree $n$, then $f + g$ is a polynomial of degree $n$.
(i) If $f$ is a polynomial of degree $n$ and $c$ is a nonzero scalar, then $cf$ is a polynomial of degree $n$.

8. In any vector space $V$ over the real numbers, show that $(a + b)(x + y) = ax + ay + bx + by$ for any $x, y \in V$ and any $a, b \in \mathbb{R}$.

12. A real-valued function $f$ defined on the real line is called an even function if $f(-t) = f(t)$ for each real number $t$. Prove that the set of even functions defined on the real line with the operations of addition and multiplication defined in Example 3:

$$ (f + g)(t) = f(t) + g(t), \quad (cf)(t) = c[f(t)], $$

is a vector space.

13. Let $V$ be denote the set of ordered pairs of real numbers. If $(a_1, a_2)$ and $(b_1, b_2)$ are elements of $V$ and $c \in \mathbb{R}$, define

$$ (a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2), \quad \text{and} \quad c(a_1, a_2) = (ca_1, ca_2). $$

Is $V$ a vector space over $\mathbb{R}$ with these operations? Justify your answer.

18. Let $V = \{(a_1, a_2) : a_1, a_2 \in \mathbb{R}\}$. For $(a_1, a_2), (b_1, b_2) \in V$ and $c \in \mathbb{R}$, define

$$ (a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 + 3b_2), \quad \text{and} \quad c(a_1, a_2) = (ca_1, ca_2). $$

Is $V$ a vector space over $\mathbb{R}$ with these operations? Justify your answer.
Section 1.3 - Pages 19-21

1. Label the following statements as true or false.
   (a) If $V$ is a vector space and $W$ is a subset of $V$ that is a vector space, then $W$ is a subspace of $V$.
   (b) The empty set is a subspace of every vector space.
   (c) If $V$ is a vector space other than the zero vector space, then $V$ contains a subspace $W$ such that $W \neq V$.
   (d) The intersection of any two subsets of $V$ is a subspace of $V$.
   (e) An $n \times n$ diagonal matrix can never have more than $n$ nonzero entries.
   (f) The trace of a square matrix is the product of its diagonal entries.
   (g) Let $W$ be the $xy$-plane in $\mathbb{R}^3$; that is $W = \{(a_1, a_2, 0) : a_1, a_2 \in \mathbb{R}\}$. Then $W = \mathbb{R}^2$.

8. Determine whether the following sets are subspaces of $\mathbb{R}^3$ under the operations of addition and scalar multiplication defined on $\mathbb{R}^3$. Justify your answers.
   (a) $W_1 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = 3a_2$ and $a_3 = -a_2\}$
   (b) $W_2 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = a_3 + 2\}$
   (d) $W_4 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 - 4a_2 - a_3 = 0\}$
   (f) $W_6 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : 5a_1^2 - 3a_2^2 + 6a_3^2 = 0\}$

12. An $m \times n$ matrix $A$ is called upper-triangular if all entries lying below the diagonal entries are zero, that is, if $A_{ij} = 0$ whenever $i > j$. Prove that the upper triangular matrices form a subspace of $M_{m \times n}(\mathbb{R})$. (You may consider only the cases $2 \times 3$, $3 \times 3$, $3 \times 2$.)

15. Is the set of all differentiable real-valued functions defined on $\mathbb{R}$ a subspace of $C(\mathbb{R})$ (the vector space of all continuous functions from $\mathbb{R}$ into $\mathbb{R}$)? Justify your answer (you may use anything learned in calculus).

21. Show that the set of convergent sequences $\{a_n\}$ (i.e. those for which $\lim_{n \to \infty} a_n$ exists) is a subspace of the vector space $V$ in Exercise 20 of Section 1.2. Namely $V$ is the set of sequences of real numbers $a_1, a_2, a_3, a_4, \ldots$, denoted $\{a_n\}$, with addition and scalar multiplication defined as follows,
   $$\{a_n\} + \{b_n\} = \{a_n + b_n\}, \quad c\{a_n\} = \{ca_n\}.$$

   With these operations $V$ is a vector space.