Department of Mathematics and Statistics
University of New Mexico

Homework 1 **Math 321: Linear Algebra** **Spring 2010**

Instructions: Attempt all problems. Only four will be graded. The exercises are in Chapter 1 of our textbook *Linear Algebra*, by S.H. Friedberg, A.J. Insel, L.E. Spence, Fourth Edition. This homework is due on Tuesday January 26, 2010.

Section 1.2 - Pages 12-15

1. Label the following statements as true or false.

   (a) Every vector space contains a zero vector.
   (b) A vector space may have more than one zero vector.
   (c) In any vector space, $ax = bx$ implies that $a = b$.
   (d) In any vector space, $ax = ay$ implies that $x = y$.
   (e) A vector in $\mathbb{R}^n$ may be regarded as a matrix in $\mathbf{M}_{n\times 1}(\mathbb{R})$.
   (f) An $m \times n$ matrix has m columns and n rows.
   (g) In $\mathbf{P}(\mathbb{R})$, only polynomials of the same degree may be added.
   (h) If f and g are polynomials of degree n, then $f + g$ is a polynomial of degree n.
   (i) If f is a polynomial of degree n and c is a nonzero scalar, then cf is a polynomial of degree n.

8. In any vector space $\mathbf{V}$ over the real numbers, show that $(a + b)(x + y) = ax + ay + bx + by$ for any $x, y \in \mathbf{V}$ and any $a, b \in \mathbb{R}$.

12. A real-valued function f defined on the real line is called an **even function** if $f(-t) = f(t)$ for each real number t. Prove that the set of even functions defined on the real line with the operations of addition and multiplication defined in Example 3:

$$(f + g)(t) = f(t) + g(t), \qquad (cf)(t) = c[f(t)],$$

is a vector space.

13. Let $\mathbf{V}$ be denote the set of ordered pairs of real numbers. If (a_1, a_2) and (b_1, b_2) are elements of $\mathbf{V}$ and $c \in \mathbb{R}$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 b_2), \quad \text{and} \quad c(a_1, a_2) = (ca_1, a_2).$$

Is $\mathbf{V}$ a vector space over $\mathbb{R}$ with these operations? Justify your answer.

18. Let $\mathbf{V} = \{(a_1, a_2) : a_1, a_2 \in \mathbb{R}\}$. For $(a_1, a_2), (b_1, b_2) \in V$ and $c \in \mathbb{R}$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 + 3b_2), \quad \text{and} \quad c(a_1, a_2) = (ca_1, ca_2).$$

Is $\mathbf{V}$ a vector space over $\mathbb{R}$ with these operations? Justify your answer.

Section 1.3 - Pages 19-21

1. Label the following statements as true or false.

   (a) If $\mathbf{V}$ is a vector space and $\mathbf{W}$ is a subset of $\mathbf{V}$ that is a vector space, then $\mathbf{W}$ is a subspace of $\mathbf{V}$.

   (b) The empty set is a subspace of every vector space.

   (c) If $\mathbf{V}$ is a vector space other than the zero vector space, then $\mathbf{V}$ contains a subspace $\mathbf{W}$ such that $\mathbf{W} \neq \mathbf{V}$.

   (d) The intersection of any two subsets of $\mathbf{V}$ is a subspace of $\mathbf{V}$.

   (e) An $n \times n$ diagonal matrix can never have more than n nonzero entries.

   (f) The trace of a square matrix is the product of its diagonal entries.

   (g) Let $\mathbf{W}$ be the xy-plane in $\mathbb{R}^3$; that is $\mathbf{W} = \{(a_1, a_2, 0) : a_1, a_2 \in \mathbb{R}\}$. Then $\mathbf{W} = \mathbb{R}^2$.

8. Determine whether the following sets are subspaces of $\mathbb{R}^3$ under the operations of addition and scalar multiplication defined on $\mathbb{R}^3$. Justify your answers.

   (a) $\mathbf{W_1} = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = 3a_2 \text{ and } a_3 = -a_2\}$

   (b) $\mathbf{W_2} = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = a_3 + 2\}$

   (d) $\mathbf{W_4} = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 - 4a_2 - a_3 = 0\}$

   (f) $\mathbf{W_6} = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : 5a_1^2 - 3a_2^2 + 6a_3^2 = 0\}$

12. An $m \times n$ matrix A is called **upper-triangular** if all entries lying below the diagonal entries are zero, that is, if $A_{ij} = 0$ whenever $i > j$. Prove that the upper triangular matrices form a subspace of $\mathbf{M}_{m\times n}(\mathbb{R})$. (You may consider only the cases 2×3, 3×3, 3×2.)

15. Is the set of all differentiable real-valued funtions defined on $\mathbb{R}$ a subspace of $C(\mathbb{R})$ (the vector space of all continuous functions from $\mathbb{R}$ into $\mathbb{R}$) ? Justify your answer (you may use anything learned in calculus).

21. Show that the set of convergent sequences $\{a_n\}$ (i.e those for which $\lim_{n\to\infty} a_n$ exists) is a subspace of the vector space $\mathbf{V}$ in Exercise 20 of Section 1.2. Namely $\mathbf{V}$ is the set of sequences of real numbers $a_1, a_2, a_3, a_4, \ldots$, denoted $\{a_n\}$, with addition and scalar multiplication defined as follows,

$$\{a_n\} + \{b_n\} = \{a_n + b_n\}, \qquad c\{a_n\} = \{ca_n\}.$$

    With these operations $\mathbf{V}$ is a vector space.