

MATH 311 - Homework #8 - SOLUTIONS

Section 2.3

p 96 #5

$$\vec{R}(t) = \cos t (\vec{i} - \vec{j}) + \sin t (\vec{i} + \vec{j}) + \frac{t}{2} \vec{k}$$

(a) $\vec{V}(t) = \vec{R}'(t) = -\sin t (\vec{i} - \vec{j}) + \cos t (\vec{i} + \vec{j}) + \frac{1}{2} \vec{k}$
 or $\vec{V}(t) = (-\sin t + \cos t) \vec{i} + (\sin t + \cos t) \vec{j} + \frac{1}{2} \vec{k}$

speed = $\frac{ds}{dt} = |\vec{V}(t)| = \sqrt{(-\sin t + \cos t)^2 + (\sin t + \cos t)^2 + \frac{1}{4}}$
 $= \sqrt{\sin^2 t - 2\sin t \cos t + \cos^2 t + \sin^2 t + 2\sin t \cos t + \cos^2 t + \frac{1}{4}}$
 $= \sqrt{1 + 1 + \frac{1}{4}} = \sqrt{\frac{4+4+1}{4}} = \sqrt{\frac{9}{4}} = \frac{3}{2} = \frac{ds}{dt}$
 SPEED

(b) Acceleration = $\vec{a}(t) = \vec{V}'(t) = \vec{R}''(t)$

$\vec{a}(t) = -\cos t (\vec{i} - \vec{j}) + \sin t (\vec{i} + \vec{j})$
 or $\vec{a}(t) = (\cos t + \sin t) \vec{i} + (\cos t + \sin t) \vec{j}$

(c) unit tangent vector = $\vec{T}(t) = \frac{\vec{R}'(t)}{|\vec{R}'(t)|} = \frac{\vec{V}(t)}{|\vec{V}(t)|} = \frac{\text{velocity}}{\text{speed}} = \frac{2}{3} \vec{V}$

$\vec{T}(t) = \frac{2}{3} \left[-\sin t (\vec{i} - \vec{j}) + \cos t (\vec{i} + \vec{j}) + \frac{1}{2} \vec{k} \right]$
 or $\vec{T}(t) = \left(-\frac{2}{3} \sin t + \frac{2}{3} \cos t \right) \vec{i} + \left(\frac{2}{3} \sin t + \frac{2}{3} \cos t \right) \vec{j} + \frac{1}{3} \vec{k}$

(d) Show curvature κ is constant.

$\vec{a} = a_t \vec{T} + a_n \vec{N}$ where $a_t = \frac{d^2s}{dt^2} = 0$ in our case

$\vec{a} = \frac{9}{4} \kappa \vec{N}$ $|\vec{N}| = 1$ $a_n = \kappa \left(\frac{ds}{dt} \right)^2 = \frac{9}{4} \kappa$
 constant curvature

$\therefore \kappa = \frac{4}{9} |\vec{a}| = \frac{4}{9} \sqrt{(-\cos t + \sin t)^2 + (\cos t + \sin t)^2} = \frac{4}{9} \sqrt{2} = \kappa$

(e) In p.75 we have the following characterization of an helix of radius ρ , with axis passing through point \vec{R}_0 , and parallel to unit vector \vec{e}_3 , given the right-handed system of unit vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$:

Helix
$$\vec{R}(t) = \vec{R}_0 + \rho \cos t \vec{e}_1 + \rho \sin t \vec{e}_2 + at \vec{e}_3$$

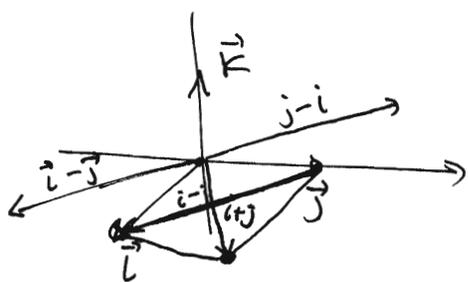
with pitch $2\pi |a|$ [right handed if $a > 0$,
left handed if $a < 0$].

In our case we have:

$$\vec{R}(t) = \cos t (\vec{i} - \vec{j}) + \sin t (\vec{i} + \vec{j}) + \frac{t}{2} \vec{k}$$

clearly $\vec{e}_3 \parallel \vec{k}$, $\vec{e}_1 \parallel \vec{i} - \vec{j}$, $\vec{e}_2 \parallel \vec{i} + \vec{j}$

For $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ To be right-handed and keep the same sign for \vec{e}_1 & \vec{e}_2



can choose

$$\vec{e}_1 = \frac{\vec{i} - \vec{j}}{|\vec{i} - \vec{j}|} = \frac{\vec{i} - \vec{j}}{\sqrt{2}}$$

$$\vec{e}_2 = \frac{\vec{i} + \vec{j}}{|\vec{i} + \vec{j}|} = \frac{\vec{i} + \vec{j}}{\sqrt{2}}$$

with this choice

$$\vec{e}_3 = -\vec{k}$$

$$\vec{R}(t) = \sqrt{2} \cos t \vec{e}_1 + \sqrt{2} \sin t \vec{e}_2 - \frac{1}{2} t \vec{e}_3$$

This is an ellipse ~~passing~~ with axis the z-axis (passing through 0), with ~~pitch~~ radius $\rho = \sqrt{2}$ and pitch $2\pi |a| = \pi$, since $a = -\frac{1}{2} < 0$ it is left handed.

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$$R(t) = \log(t^2+1) \vec{i} + (t - 2 \arctan t) \vec{j} + 2\sqrt{2}t \vec{k}$$

(a) $V(t) = R'(t) = \frac{2t}{t^2+1} \vec{i} + \left(1 - \frac{2}{1+t^2}\right) \vec{j} + 2\sqrt{2} \vec{k}$

$$\begin{aligned} \text{Speed} &= |V(t)| = \sqrt{\frac{4t^2}{(t^2+1)^2} + \left(\frac{1+t^2-2}{1+t^2}\right)^2 + 4 \cdot 2} \\ &= \sqrt{\frac{4t^2 + (t^2-1)^2 + 8(1+t^2)^2}{1+t^2}} \\ &= \sqrt{\frac{4t^2 + (t^4 - 2t^2 + 1) + 8(1 + 2t^2 + t^4)}{1+t^2}} \\ &= \frac{\sqrt{9t^4 + 18t^2 + 9}}{1+t^2} = \frac{\sqrt{9(t^4 + 2t^2 + 1)}}{1+t^2} = 3 \frac{\sqrt{(t^2+1)^2}}{t^2+1} \end{aligned}$$

$$\frac{ds}{dt} = 3 \frac{(t^2+1)}{t^2+1} = 3 \leftarrow \text{constant speed}$$

(b) Find curvature κ .

acceleration $\vec{a}(t) = a_t \vec{T} + a_n \vec{N}$

where $a_t = \frac{ds}{dt} = 0$
since speed is constant

$$a_n = \left(\frac{ds}{dt}\right)^2 \kappa = 9\kappa$$

$$\rightarrow 9\kappa \vec{N} = |\vec{a}(t)| \vec{a}$$

$$\boxed{\kappa = \frac{1}{9} |\vec{a}(t)|}$$

Now $a(t) = v'(t) = \frac{2(t^2+1) - 2t(2t)}{(t^2+1)^2} \vec{i} + \left(\frac{-2(2t)}{(1+t^2)^2}\right) \vec{j} = \frac{-2t^2+2}{(t^2+1)^2} \vec{i} - \frac{4t}{(1+t^2)^2} \vec{j}$

$$|a(t)| = \sqrt{\frac{4(1-t^2)^2}{(1+t^2)^4} + \frac{16t^2}{(1+t^2)^4}} = \frac{\sqrt{4-8t^2+4t^4+16t^2}}{(1+t^2)^2} = \frac{\sqrt{4(t^4+2t^2+1)}}{(1+t^2)^2} = \frac{2(t^2+1)}{(1+t^2)^2}$$

$$|a(t)| = \frac{2}{1+t^2} \Rightarrow \boxed{\kappa = \frac{2}{9(1+t^2)}}$$