

Sec 5.5

p. 299 #1

$$\mathbf{F} = 3y \vec{i} + (5-2x) \vec{j} + (z^2-2) \vec{k}$$

$$(a) \operatorname{Div} \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 0 + 0 + 2z = 2z \quad [2z]$$

$$(b) \operatorname{curl} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3y & 5-2x & z^2-2 \end{vmatrix} = +(-2-3)\vec{k} = [-5\vec{k}]$$

$$(c) \iint_S \operatorname{curl} \vec{F} \cdot \vec{n} dS = \iint_S \left(-5 \vec{k} \right) \cdot \frac{(x\vec{i} + y\vec{j} + z\vec{k})}{2} \cdot \vec{n} dS$$

$S: x^2 + y^2 + z^2 = 4$ $\vec{n} = \left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2} \right)$ $\vec{n} \cdot \vec{k} > 0$
 $\iint_S -\frac{5z}{2} dS \quad (*)$

Long Way
 Using spherical coordinates
 $r=2$ $0 \leq \theta \leq 2\pi$ $0 \leq \phi \leq \frac{\pi}{2}$

$$dS = r^2 \sin \phi d\phi d\theta = 4 \sin \phi d\phi d\theta$$

$$\begin{aligned} \iint_S \operatorname{curl} \vec{F} \cdot \vec{n} dS &= \frac{1}{2} \int_0^{2\pi} \int_0^{\pi/2} -10 \cos \phi \cdot 4 \sin \phi d\phi d\theta \\ &= -20 \int_0^{2\pi} \left[-\frac{\cos(2\phi)}{2} \right]_{\phi=0}^{\pi/2} d\theta = -20 \int_0^{2\pi} \frac{-\cos \pi + \cos 0}{2} d\theta \\ &= -20 \frac{(2\pi)}{2} = \boxed{-40\pi} = \boxed{-20\pi} \end{aligned}$$

(Can't see now how to reduce to a triviality as advertised in the problem
 I KNOW Ü

P-299 #1 (cont)

Here is the easy proof.

Consider the CLOSED surface ~~S consisting~~ bounded by the hemisphere $x^2 + y^2 + z^2 = 4 \quad z > 0$ and the plane $z = 0$



$$\iint_{S^*} \text{curl } \vec{F} \cdot d\vec{s} = \iiint_{\text{solid bounded by } S^*} \text{div}(\text{curl } \vec{F}) dv$$

$$\text{but } \text{curl } \vec{F} = -5 \vec{k} \text{ constant}$$

$$\text{so } \text{div}(\text{curl } \vec{F}) = 0$$

$$0 = \iint_{S^*} \text{curl } \vec{F} \cdot d\vec{s} = \iint_S \text{curl } \vec{F} \cdot d\vec{s} + \iint_{\text{Area}} (\text{curl } \vec{F} \cdot \vec{k}) dS$$

$$\therefore \iint_S \text{curl } \vec{F} \cdot d\vec{s} = \iint_{\text{Area}} -5 \vec{k} \cdot \vec{k} dS = -5 \text{Area}$$



$$= \cancel{\pi}(-5) \pi(2)^2$$

$$= \boxed{-20\pi}$$

Sec 5.5

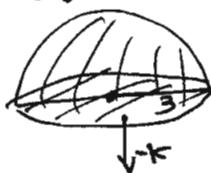
$$\operatorname{curl} \vec{F} = 2y \vec{i} - 2z \vec{j} + 3k$$

p.299 #2

$$\text{or } \oint_S \vec{F} \cdot d\vec{s} = \iint_{S^*} \vec{F} \cdot \vec{n} \, dS = \iiint_V dV (\operatorname{curl} \vec{F}) = 0$$

Since $\iint_V (\operatorname{curl} \vec{F}) = 0$ → closed surface

(a) Let S^* be the closed surface consisting of hemispherical surface $x^2 + y^2 + z^2 = 9, z > 0$ and the circle $x^2 + y^2 = 9, z = 0$



$$\text{Then } 0 = \iint_{S^*} \vec{F} \cdot \vec{n} \, dS = \iint_S \vec{F} \cdot \vec{n} \, dS + \iint_{\text{bottom}} \vec{F} \cdot \vec{k} \, dS$$



$$\Rightarrow \iint_S \vec{F} \cdot \vec{n} \, dS = - \iint_{\text{bottom}} 3k \cdot (-k) \, dS = +3 \text{ Area } \text{circle}_3 = \\ = 3\pi 3^2 = 27\pi$$

(b) By the argument at the beginning,

$$\iint_S \vec{F} \cdot d\vec{s} = 0$$



closed sphere

p.299 #3

Prove $\iint_S (\nabla \phi \times \nabla \psi) \cdot d\vec{s} = \oint_C \phi \nabla \psi \cdot d\vec{R}$

by Stoke's Thm, suffices to verify that
but this is (3.29) in p147
 $\vec{F} = \nabla \psi + \text{The observation}$
 $\nabla \times (\phi \nabla \psi) = \nabla \phi \times \nabla \psi$
 $\nabla \times \nabla \psi = 0$.

Schaum's
p.134 #67]

Show that

\vec{B} is normal to Surface S'

$$\iiint_V \text{curl } \vec{B} \, dV = \vec{0} \quad (*)$$

$$\vec{B} = (B_1, B_2, B_3)$$

$$\text{curl } \vec{B} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_1 & B_2 & B_3 \end{vmatrix} = \left(\frac{\partial B_3}{\partial y} - \frac{\partial B_2}{\partial z} \right) \vec{i} - \left(\frac{\partial B_3}{\partial x} - \frac{\partial B_1}{\partial z} \right) \vec{j} + \left(\frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y} \right) \vec{k}$$

(*) means that each one of

The volume integrals $\iiint_V \left(\frac{\partial B_3}{\partial y} - \frac{\partial B_2}{\partial z} \right) dV = 0, \text{ etc.}$

we can view $\frac{\partial B_3}{\partial y} - \frac{\partial B_2}{\partial z} = \vec{dN} \cdot \vec{F}$

for $\vec{F} = B_3 \vec{j} - B_2 \vec{k}$ so by divergence theorem

$$\iiint_V \vec{dN} \cdot \vec{F} \, dV = \iint_S \vec{F} \cdot \vec{dS} = \iint_S (B_3 \vec{j} - B_2 \vec{k}) \cdot \vec{n} \, dS = 0$$

what we knew was that $(B_1, B_2, B_3) \parallel \vec{n}$

so can write $\vec{n} = \frac{(B_1, B_2, B_3)}{\sqrt{B_1^2 + B_2^2 + B_3^2}}$

$$\vec{F} \cdot \vec{n} = (0, B_3, -B_2) \cdot \frac{(B_1, B_2, B_3)}{\sqrt{B_1^2 + B_2^2 + B_3^2}} = \frac{B_3 B_2 - B_2 B_3}{\sqrt{B_1^2 + B_2^2 + B_3^2}} = 0$$

Similarly for the other 2 integrals.

Schau'm's

p. 134 #68

$$\int_C \mathbf{E} \cdot d\mathbf{r} = -\frac{1}{c} \frac{\partial H}{\partial t} \iint_S \vec{H} \cdot d\vec{S}$$

S is an orientable surface bounded by C . Anti orientation of S determines orientation on C .

Show that

$$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{H}}{\partial t}$$

Stoke's thm gives:

$$\begin{aligned} \int_C \mathbf{E} \cdot d\mathbf{r} &= \iint_S (\nabla \times \mathbf{E}) \cdot d\vec{S} = -\frac{1}{c} \frac{\partial H}{\partial t} \iint_S \vec{H} \cdot d\vec{S} \\ &= \iint_S \left(-\frac{1}{c} \frac{\partial \mathbf{H}}{\partial t}\right) \cdot d\vec{S} \end{aligned}$$

Since this holds for any such surface, we can let the surface shrink to a point, dividing by the surface area. In the limit we will get:

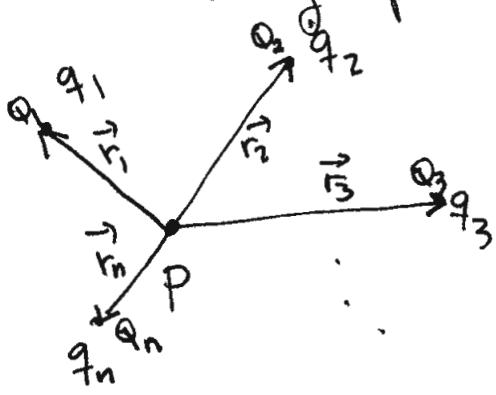
$$\begin{aligned} \boxed{(\nabla \times \mathbf{E}) \cdot \vec{n}} &= \lim_{S \rightarrow \text{pt}} \frac{1}{\text{Surface Area}(S)} \iint_S (\nabla \times \mathbf{E}) \cdot \vec{n} dS \\ &= \lim_{S \rightarrow \text{pt}} \frac{1}{\text{Surface Area}(S)} \iint_S \left(-\frac{1}{c} \frac{\partial \mathbf{H}}{\partial t}\right) \cdot d\vec{S} \\ &= \boxed{-\frac{1}{c} \frac{\partial \mathbf{H}}{\partial t} \cdot \vec{n}} \end{aligned}$$

$$\Rightarrow \boxed{\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{H}}{\partial t}} \quad \text{as requested}$$

Schaum's

p. 134 #73

The potential $\phi(P)$ at a point $P(x_1, y_1, z_1)$ due to a system of charges (masses) q_1, q_2, \dots, q_n having position vectors $\vec{r}_1, \vec{r}_2, \dots, \vec{r}_n$ w.r.t P



is given by:

$$\phi = \sum_{m=1}^n \frac{q_m}{r_m} \quad r_m = |\vec{r}_m|$$

Prove Gauss' Law:

$$\iint_S \vec{E} \cdot d\vec{S} = 4\pi Q$$

where $E = -\nabla\phi$ is the electric field intensity,
 S is a surface enclosing all charges and
 $Q = \sum_{m=1}^n q_m$ is the total charge within S

Problem #3 in Exam 2 was Gauss Law for one point.

One can calculate

$$\vec{E} = \sum_{m=1}^n \frac{q_m \vec{r}_m}{|\vec{r}_m|^3}$$

defined everywhere except at

$$\vec{E}(P) = \sum_{m=1}^n q_m \frac{(Q_m - P)}{|Q_m - P|^3} \quad q_1, \dots$$

One can verify that

$\nabla \cdot \vec{E} = 0$, so if we consider the little balls B_1, \dots, B_n centered

region R inside S, and outside at each of the points Q_1, \dots, Q_n , Then

$$0 = \iiint_R \nabla \cdot \vec{E} dV = \iint_S \vec{E} \cdot d\vec{S} - \iint_{B_1} \vec{E} \cdot d\vec{S} - \dots - \iint_{B_n} \vec{E} \cdot d\vec{S}$$

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Schawm p134 #73 (cont)

So

$$\iint_S \vec{E} \cdot d\vec{s} = \sum_{m=1}^n \iint_{B_m} \vec{E} \cdot d\vec{s}$$

We choose the balls B_1, \dots, B_m small enough so that the only charge inside each one of them is the one corresponding to its center.

$$\iint_{B_m} \vec{E} \cdot d\vec{s} = \sum_{k=1}^n q_k \underbrace{\iint_{B_m} \frac{(q_k - p)}{|q_k - p|^3} \cdot d\vec{s}}_{\begin{cases} 0 & \text{if } k \neq m \\ 4\pi & \text{if } k = m \end{cases}}$$

by the
calculation
in Exam 2 #3(b)

$$\therefore \iint_S \vec{E} \cdot d\vec{s} = \sum_{m=1}^n 4\pi q_m = 4\pi \left(\sum_{m=1}^n q_m \right) = 4\pi Q$$

as requested