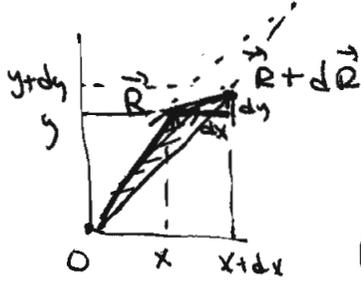


Section 5.4

p 294 #5

$$\vec{R} = x\vec{i} + y\vec{j} \quad dR = dx\vec{i} + dy\vec{j}$$

(a) Compute  $|R \times (R+dR)| = \begin{vmatrix} i & j & k \\ x & y & 0 \\ x+dx & y+dy & 0 \end{vmatrix} = \left| (xy+xdy-yx-ydk) \right|$   
 $= (x dy - y dx) |k| = \boxed{x dy - y dx}$



(b)  $\frac{x dy - y dx}{2} = \frac{|R \times (R+dR)|}{2} = \frac{1}{2}$  Area of the parallelogram determined by  $\vec{R}$  &  $\vec{R}+d\vec{R}$   
 $=$  Area of the  $\Delta$  with vertices at  $(0,0)$ ,  $(x,y)$  and  $(x+dx, y+dy)$

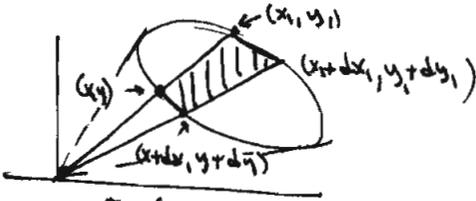
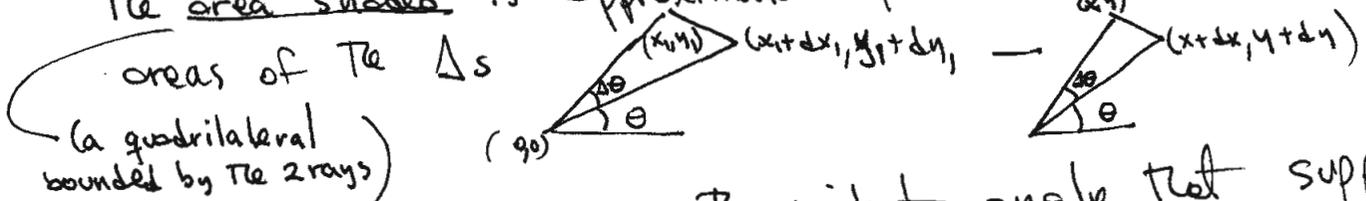


Figure 5.6

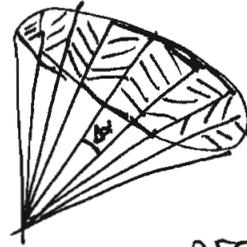
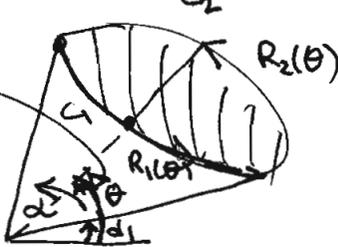
(c) For a domain D that has the property that any ray intersects the boundary at at most 2 points, like in the picture then the area shaded is approximated by the difference of the areas of the  $\Delta$ s



(a quadrilateral bounded by the 2 rays)

Given a partition of the widest angle that supports the domain

into  $n$  smaller angles  $\Delta\theta = \frac{\alpha}{n}$



Then the area of the domain is approximated by the sum of the quadrilaterals' areas. If we use the parameter  $\theta$  to parametrize both the domain and the curve,

$\rightarrow R_1(\theta)$   $\theta: \alpha + d_1 \rightarrow \alpha_1$  Domain  $\alpha_1 \leq \theta \leq \alpha_1 + \alpha$   
 $R_2(\theta)$   $\theta: \alpha_1 \rightarrow \alpha + d_1$   $R_1(\theta) \leq r \leq R_2(\theta)$  (polar coordinates)

as  $\Delta\theta \rightarrow d\theta$  Area =  $\int_{C_2} \frac{x dy - y dx}{2} - \int_{C_1} \frac{x dy - y dx}{2}$   
 $= \int_C \frac{x dy - y dx}{2} = \text{Area of } D.$

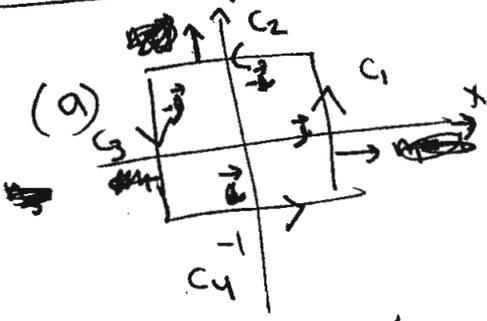
P.294 #7

$C: x^2 + y^2 = 9$   $\vec{F} = y\vec{i} - 3x\vec{j}$   $\frac{\partial F_1}{\partial y} = 1, \frac{\partial F_2}{\partial x} = -3$

$\oint_C \vec{F} \cdot d\vec{R} = \iint_D \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx dy = \iint_D (-3 - 1) dS = -4 \text{ Area}$   
 by Green's Theorem   
 $= -4\pi(3)^2 = -36\pi$

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$\int_C (4y^3 dx - 2x^2 dy)$  around square bounded by  $x = \pm 1, y = \pm 1$



4 sides to consider

$C_2$   $\vec{R} = (x, 1)$   $x: -1 \rightarrow +1$   
 $\vec{R}' = (1, 0)$

$\int_{C_2} (4y^3, -2x^2) \cdot (1, 0) dx = \int_{-1}^{+1} -2x^2 dy = \int_{-1}^{+1} -2x^2 dx = -2 \cdot 2 = -4$   
 $\int_{C_2} (4y^3, -2x^2) \cdot (1, 0) dx = \int_{-1}^{+1} 4 dx = 4 \cdot 2 = 8$

$C_3$   $\vec{R} = (-1, y)$   $y: 1 \rightarrow -1$   
 $\vec{R}' = (0, 1)$

$\int_{C_3} (4y^3, -2x^2) \cdot (0, 1) dy = \int_{1}^{-1} -2 dy = (-2)(-2) = 4$

$C_4$   $\vec{R} = (x, -1)$   $x: -1 \rightarrow 1$   
 $\vec{R}' = (1, 0)$

$\int_{C_4} (4y^3, -2x^2) \cdot (1, 0) dx = \int_{-1}^1 -4 dx = -4(2) = -8$

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#9 (cont)

(a)  $\int_C 4y^3 dx - 2x^2 dy = -4 - 8 + 4 - 8 = \boxed{-16}$

(b) Green's Theorem says

$$\int_C F_1 dx + F_2 dy = \iint_R \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx dy$$

In our case

$$F_1 = 4y^3 \rightarrow \frac{\partial F_1}{\partial y} = 12y^2$$

$$\Rightarrow \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = -4x - 12y^2$$

$$F_2 = -2x^2 \rightarrow \frac{\partial F_2}{\partial x} = -4x$$

$$\therefore \int_C 4y^3 dx - 2x^2 dy = \int_{-1}^1 \int_{-1}^1 (-4x - 12y^2) dx dy =$$

$$= \int_{-1}^1 \left( -4 \frac{x^2}{2} - 12y^2 x \right) \Big|_{x=-1}^{x=1} dy = \int_{-1}^1 \left[ -2 - 12y^2 - (-2 + 12y^2) \right] dy$$

$$= \int_{-1}^1 -24y^2 dy = -24 \cdot \frac{y^3}{3} \Big|_{-1}^1 = \frac{-24 - (-24)}{3} = \frac{-48}{3} = \boxed{-16}$$

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#12

Use Green's Theorem to find area inside the

loop of Descartes folium

$$x = \frac{t}{1+t^3}, \quad y = \frac{t^2}{1+t^3}$$

$$0 \leq t < \infty$$

$$\text{Area} = \frac{1}{2} \int (x dy - y dx),$$

p. 295 #12 (cont)

$$x = \frac{t}{1+t^3}, \quad y = \frac{t^2}{1+t^3}$$

$$dx = \frac{1(1+t^3) - t(3t^2)}{(1+t^3)^2} = \frac{1-2t^3}{(1+t^3)^2} \quad 0 \leq t < \infty$$

$$dy = \frac{2t(1+t^3) - t^2(3t^2)}{(1+t^3)^2} = \frac{2t-t^4}{(1+t^3)^2}$$

$$\therefore \text{Area} = \frac{1}{2} \int x dy - y dx = \frac{1}{2} \int_0^{\infty} \frac{t}{1+t^3} \cdot \frac{(2t-t^4)}{(1+t^3)^2} - \frac{t^2(1-2t^3)}{(1+t^3)^3} dt$$

$$= \frac{1}{2} \int_0^{\infty} \frac{2t^2 - t^5 - t^2 + 2t^5}{(1+t^3)^3} dt = \frac{1}{2} \int_0^{\infty} \frac{t^2 + t^5}{(1+t^3)^3} dt$$

$$= \frac{1}{2} \int_0^{\infty} \frac{t^2(1+t^3)}{(1+t^3)^3} dt = \frac{1}{2} \int_0^{\infty} \frac{t^2}{(1+t^3)^2} dt$$

subst

$$\begin{aligned} u &= 1+t^3 \\ du &= 3t^2 dt \\ t=0 &\rightarrow u=1 \\ t=\infty &\rightarrow u=\infty \end{aligned}$$

$$= \frac{1}{2} \cdot \frac{1}{3} \int_1^{\infty} \frac{du}{u^2} = \frac{1}{6} \left( \frac{-1}{u} \right) \Big|_1^{\infty}$$

$$= \frac{1}{6} (0 + 1) = \boxed{\frac{1}{6}}$$