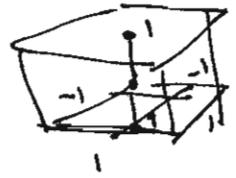


Section 4.9

p. 262 #4

Find  $\iiint_V \text{div } F \, dV$  for  $F = (x^2 + xy)\vec{i} + (y^2 + yz)\vec{j} + (z^2 + zx)\vec{k}$

and  $V$  cube centered at origin with faces on planes  $x = \pm 1, y = \pm 1, z = \pm 1$



$-1 \leq x \leq 1$   
 $-1 \leq y \leq 1$   
 $-1 \leq z \leq 1$

$\text{div } F = 2x + y + 2y + z + 2z + x$   
 $\text{div } F = 3(x + y + z)$

$$\begin{aligned} \iiint_V \text{div } F \, dV &= 3 \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 (x + y + z) \, dx \, dy \, dz = 3 \int_{-1}^1 \int_{-1}^1 \left( \frac{x^2}{2} + (y+z)x \right) \Big|_{-1}^1 \, dy \, dz \\ &= 3 \int_{-1}^1 \int_{-1}^1 \left( \frac{1}{2} + (y+z) - \left( \frac{1}{2} - (y+z) \right) \right) \, dy \, dz \\ &= 6 \int_{-1}^1 \int_{-1}^1 (y+z) \, dy \, dz = 6 \int_{-1}^1 \left( \frac{y^2}{2} + zy \right) \Big|_{-1}^1 \, dz \\ &= 6 \int_{-1}^1 \left( \frac{1}{2} + z - \left( \frac{1}{2} - z \right) \right) \, dz = 12 \int_{-1}^1 z \, dz = 12 \left. \frac{z^2}{2} \right|_{-1}^1 = \boxed{0} \end{aligned}$$

Had we used the divergence theorem, we would have noticed that on the faces where  $x=1$   $x=-1$  normals are  $i$  and  $-i$  respectively,  $y=\pm 1$   $n=\pm j$   $z=\pm 1$   $n=\pm k$

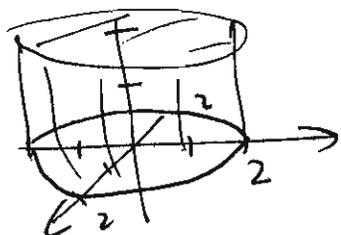
$x=1$	$F \cdot n = (1+y)\vec{i} \cdot \vec{i} = 1+y$	} $\int_{-1}^1 \int_{-1}^1 2y \, dz \, dy = 2 \cdot 2 \left. \frac{y^2}{2} \right _{-1}^1 = 0$
$x=-1$	$F \cdot n = (1-y)\vec{i} \cdot (-\vec{i}) = -1+y$	
$y=1$	$F \cdot n = (1+z)\vec{j} \cdot \vec{j} = 1+z$	} $\int_{-1}^1 \int_{-1}^1 2z \, dx \, dz = 2 \cdot 2 \left. \frac{z^2}{2} \right _{-1}^1 = 0$
$y=-1$	$F \cdot n = (1-z)\vec{j} \cdot (-\vec{j}) = -1+z$	
$z=1$	$F \cdot n = (1+x)\vec{k} \cdot \vec{k} = 1+x$	} $\int_{-1}^1 \int_{-1}^1 2x \, dy \, dx = 2 \cdot 2 \left. \frac{x^2}{2} \right _{-1}^1 = 0$
$z=-1$	$F \cdot n = (1-x)\vec{k} \cdot (-\vec{k}) = -1+x$	

$\boxed{= 0}$

**p262 #5** Use div. thm to evaluate  $\iint_S \mathbf{F} \cdot \mathbf{n} \, dS$

when  $\mathbf{F} = y^2 x \vec{i} + x^2 y \vec{j} + z^2 \vec{k}$

$S$  is the <sup>complete</sup> surface bounded by cylinder  $x^2 + y^2 = 4$  and planes  $z=0, z=2$



$\vec{F}$  is continuous and diff. in  $\mathbb{R}^3$

by div. thm  $\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_V \text{div } \mathbf{F} \, dV$

$\text{div } \mathbf{F} = y^2 + x^2 + z^2$

In cylindrical coordinates  $x^2 + y^2 = \rho^2$

$0 \leq \rho \leq 2$

$0 \leq \theta \leq 2\pi$

$0 \leq z \leq 2$

$\text{div } \mathbf{F} = \rho^2 + z^2$

$\iiint_V \text{div } \mathbf{F} \, dV = \int_0^2 \int_0^{2\pi} \int_0^2 (\rho^2 + z^2) \rho \, d\rho \, d\theta \, dz$

$= \int_0^2 \int_0^{2\pi} \int_0^2 (\rho^3 + z^2 \rho) \, d\rho \, d\theta \, dz$

$= \int_0^2 \int_0^{2\pi} \left( \frac{\rho^4}{4} + z^2 \frac{\rho^2}{2} \right) \Big|_0^2 \, d\theta \, dz$

$= \int_0^2 \int_0^{2\pi} (4 + 2z^2) \, d\theta \, dz = 2\pi \int_0^2 (4 + 2z^2) \, dz$

$= 2\pi \left( 4z + \frac{2z^3}{3} \right) \Big|_0^2 = 2\pi \left( 8 + \frac{16}{3} \right)$

$= \frac{2\pi}{3} (24 + 16) = \boxed{\frac{80\pi}{3}} = \iint_S \mathbf{F} \cdot \mathbf{n} \, dS$

section 4.9  
p. 263 #10

Find  $\int F \cdot dR$  around ellipse  $x^2 + y^2 = 1$

and  $z = y$ , where  $F = x\vec{i} + (x+y)\vec{j} + (x+y+z)\vec{k}$

Line integral directly

$x = \cos \theta$   
 $y = \sin \theta$   
 $z = \sin \theta$   
 $0 \leq \theta \leq 2\pi$

$dR = (-\sin \theta, \cos \theta, \cos \theta) d\theta$

$\int F \cdot dR = \int_0^{2\pi} (-\cos \theta \sin \theta + \cos^2 \theta + \cos \theta \sin \theta - \sin \theta \cos \theta) d\theta$   
 $= \int_0^{2\pi} (2\cos \theta \sin \theta + \cos 2\theta - 1) d\theta$

⌚

$= \int_0^{2\pi} (2\cos \theta \sin \theta + \cos 2\theta - 1) d\theta$

$u = \cos \theta \quad du = -\sin \theta d\theta$   
 $= -\int_1^{-1} u du + (-\frac{\sin 2\theta}{2} - \theta) \Big|_0^{2\pi}$

$= (-2\pi) = \boxed{-2\pi}$

Using Stoke's Thm

$\int_C \vec{F} \cdot d\vec{R} = \iint_S \text{curl } \vec{F} \cdot d\vec{S}$

$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & x+y & x+y+z \end{vmatrix} = \vec{i} - \vec{j} + \vec{k}$

plane  $-y+z=0 \quad \vec{n} = \frac{(0, -1, 1)}{\sqrt{2}}$

$\text{curl } \vec{F} \cdot \vec{n} = \frac{1}{\sqrt{2}} (1+1) = \frac{2}{\sqrt{2}} = \sqrt{2}$

$= \iint_S \text{curl } \vec{F} \cdot \vec{n} \frac{dS}{\cos \gamma} =$

$= \iint_{x^2+y^2=1} \sqrt{2} \frac{dx dy}{\frac{\sqrt{2}}{2}}$   
 $\cos \gamma = \frac{\sqrt{2}}{2}$

$= 2 \text{ Area Circle } (x^2+y^2=1)$

$= 2 \cdot \pi \cdot 1^2 = \boxed{2\pi}$

The choice of normal surface opposite orientation of C

Section 4.9

p. 263 #13

Evaluate  $\iint_S (\nabla \times F) \cdot n \, dS$  where

$$\vec{F} = 2y \vec{i} + (x - 2x^3 z) \vec{j} + xy^3 \vec{k}$$

where  $S$  is the curved surface of the hemisphere

$$x^2 + y^2 + z^2 = 1 \quad z \geq 0$$



By Stokes' Theorem

$$\iint_S (\nabla \times F) \cdot n \, dS = \int_C F \cdot dR$$

$$C: x^2 + y^2 = 1$$

on  $z=0$

$$x = \cos \theta \quad 0 \leq \theta \leq 2\pi$$

$$y = \sin \theta$$

$0 \leq \theta \leq 2\pi$  (correct orientation)

$$dR = (-\sin \theta, \cos \theta, 0) d\theta$$

$$= \int_0^{2\pi} (2 \sin \theta, \cos \theta, \cos \theta \sin^3 \theta) \cdot (-\sin \theta, \cos \theta, 0) d\theta$$

$$= \int_0^{2\pi} (-2 \sin^2 \theta + \cos^2 \theta) d\theta$$

$$= \int_0^{2\pi} (1 - 3 \sin^2 \theta) d\theta$$

$$= \int_0^{2\pi} \left( 1 + \frac{3}{2} (1 - 2 \sin^2 \theta) - \frac{3}{2} \right) d\theta$$

$$= \int_0^{2\pi} \left( -\frac{1}{2} + \frac{3}{2} \cos 2\theta \right) d\theta$$

$$= -\frac{\theta}{2} + \frac{3}{2} \left( -\frac{\sin 2\theta}{2} \right) \Big|_0^{2\pi}$$

$$= -\frac{2\pi}{2} = \boxed{-\pi}$$

p. 263 #13

We could have calculated the surface

integral directly.

$$\nabla_x F = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y & (x-2x^3z) & xy^3 \end{vmatrix} = \begin{matrix} (3xy^2 + 2x^3) \mathbf{i} \\ -(y^3 - 0) \mathbf{j} \\ + (1 - 6x^2z - 2) \mathbf{k} \end{matrix}$$

$$\nabla_x F = (3xy^2 + 2x^3) \mathbf{i} - y^3 \mathbf{j} - (6x^2z + 1) \mathbf{k}$$

In spherical coordinates :  $x = \sin \phi \cos \theta$   $z = \cos \phi$   
 $y = \sin \phi \sin \theta$

$r = 1$

$dS = \sin \phi \, d\phi \, d\theta$

$0 \leq \phi \leq \frac{\pi}{2}$

$0 \leq \theta \leq 2\pi$

$\vec{n} = (x, y, z) = (\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi)$

$(\nabla_x F \cdot \vec{n}) = 3x^2y^2 + 2x^4 - y^4 - 6x^2z^2 - z$   
 $= 3 \sin^4 \phi \cos^2 \theta \sin^2 \theta + 2 \sin^4 \phi \cos^4 \theta - \sin^4 \phi \sin^2 \theta - 6 \sin^2 \phi \cos^2 \phi \cos^2 \theta$   
 $= 6 \cos^2 \phi$

Finally  $\int_0^{\pi/2} \int_0^{2\pi} (\nabla_x F \cdot \vec{n}) \sin \phi \, d\theta \, d\phi =$

$= \int_0^{\pi/2} \int_0^{2\pi} \left[ \sin^5 \phi (3 \cos^2 \theta \sin^2 \theta + 2 \cos^4 \theta - \sin^2 \theta) - 6 \sin^3 \phi \cos^2 \phi \cos^2 \theta + \sin \phi \cos \phi \right] d\theta \, d\phi$   
 $\quad \quad \quad \cos^2 \phi (1 - \cos^2 \phi) \sin \phi \quad \rightarrow \frac{\sin(2\phi)}{2}$

$\phi = 0 \Rightarrow u = 1$   
 $\phi = 2\pi \Rightarrow u = 1$   
 $u = \cos \phi$   
 $du = -\sin \phi \, d\phi$

$= \int_0^{\pi/2} \left( 2 \cos^2 \theta (\sin^2 \theta + \cos^2 \theta) + \sin^2 \theta (\cos^2 \theta - 1) \right) \int_1^1 \frac{(1-u^2)^2}{(-du)} \, d\phi$   
 Integrates to 0

$\int_0^{\pi/2} 6 \cos^2 \theta \int_1^1 (u^2(1-u^2))(-du) = 0$   
 uups...  
 Mistake somewhere