

Section 3.10

p. 170 #6

Compute $\nabla^2 f$ in cylindrical and spherical coordinates (Hint: use $\nabla^2 = \operatorname{div} \operatorname{grad}$)

Cylindrical coordinates: $f = f(\rho, \theta, z)$, $\mathbf{F} = F_\rho \mathbf{e}_\rho + F_\theta \mathbf{e}_\theta + F_z \mathbf{e}_z$

$$\text{Remember } \left\{ \begin{array}{l} \operatorname{grad} f = \frac{\partial f}{\partial \rho} \mathbf{e}_\rho + \frac{1}{\rho} \frac{\partial f}{\partial \theta} \mathbf{e}_\theta + \frac{\partial f}{\partial z} \mathbf{e}_z \\ \operatorname{div} \mathbf{F} = \frac{1}{\rho} \frac{\partial (\rho F_\rho)}{\partial \rho} + \frac{1}{\rho} \frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z} \end{array} \right. \quad (3.55) \quad (3.58)$$

In our case $\mathbf{F} = \operatorname{grad} f$, $F_\rho = \frac{\partial f}{\partial \rho}$, $F_\theta = \frac{1}{\rho} \frac{\partial f}{\partial \theta}$, $F_z = \frac{\partial f}{\partial z}$

so ~~$\nabla^2 f = \operatorname{div} \mathbf{F} =$~~ $\nabla^2 f = \frac{1}{\rho} \frac{\partial (\rho \frac{\partial f}{\partial \rho})}{\partial \rho} + \frac{1}{\rho} \frac{\partial \left(\frac{1}{\rho} \frac{\partial f}{\partial \theta} \right)}{\partial \theta} + \frac{\partial^2 f}{\partial z^2}$

$$\boxed{\nabla^2 f = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial f}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}}$$

Spherical coordinates $f = f(r, \phi, \theta)$, $\mathbf{F} = F_r \mathbf{e}_r + F_\phi \mathbf{e}_\phi + F_\theta \mathbf{e}_\theta$

$$\text{Remember } \left\{ \begin{array}{l} \operatorname{grad} f = \frac{\partial f}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial f}{\partial \phi} \mathbf{e}_\phi + \frac{1}{r \sin \phi} \frac{\partial f}{\partial \theta} \mathbf{e}_\theta \\ \operatorname{div} \mathbf{F} = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 F_r \right) + \frac{1}{r \sin \phi} \frac{\partial}{\partial \phi} \left(F_\phi \sin \phi \right) + \frac{1}{r \sin \phi} \frac{\partial F_\theta}{\partial \theta} \end{array} \right.$$

so $\nabla^2 f = \operatorname{div} (\operatorname{grad} f) = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r \sin \phi} \frac{\partial}{\partial \phi} \left(\frac{1}{r} \frac{\partial f}{\partial \phi} \right) + \frac{1}{r \sin \phi} \frac{\partial}{\partial \theta} \left(\frac{1}{r \sin \phi} \frac{\partial f}{\partial \theta} \right)$

$$\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin^2 \phi} \frac{\partial}{\partial \phi} \left(\frac{\partial f}{\partial \phi} \right) + \frac{1}{r^2 \sin^2 \phi} \frac{\partial^2 f}{\partial \theta^2}$$

$$\boxed{\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin^2 \phi} \frac{\partial}{\partial \phi} \left(\sin \phi \frac{\partial f}{\partial \phi} \right) + \frac{1}{r^2 \sin^2 \phi} \frac{\partial^2 f}{\partial \theta^2}}$$

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p 170 #7

Then

(Bonus in mid term).

Show that if f is a function of r only

$$\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$$

Using formula \circledast for the Laplacian in spherical coordinates found in problem #6, we see

that since $\frac{\partial f}{\partial \phi} = 0$, $\frac{\partial f}{\partial \theta} = 0$ (because f depends only on r)

$$\begin{aligned}\nabla^2 f &= \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \phi} \frac{\partial}{\partial \phi} (0) + \frac{1}{r^2 \sin^2 \phi} (0) \\ &= \frac{1}{r^2} \left[2r^2 \frac{\partial^2 f}{\partial r^2} + 2r \frac{\partial f}{\partial r} \right]\end{aligned}$$

$\boxed{\nabla^2 f = f'' + \frac{2}{r} f'}$

p. 170 #8(b)

Change to cylindrical coordinates and find divergence and curl of

$$F = -y \vec{i} + x \vec{j} = -\frac{p \sin \theta \vec{i} + p \cos \theta \vec{j}}{r^2}$$

$$= +\frac{1}{r} \left[-\sin \theta \vec{i} + \cos \theta \vec{j} \right] = \frac{1}{r} e_\theta$$

Use (3.58) to find the divergence: $F_e = 0$, $F_\theta = \frac{1}{r} e_\theta$, $F_z = 0$

$$\text{div } F = \frac{1}{r} \left(\cancel{\frac{\partial (r F_e)}{\partial r}} + \frac{1}{r} \cancel{\frac{\partial F_\theta}{\partial \theta}} + \cancel{\frac{\partial F_z}{\partial z}} \right) = \frac{1}{r} \cancel{\frac{\partial}{\partial \theta} \left(\frac{1}{r} \right)} = 0$$

| P. 170 # 8(b) (cont)

Similarly compute $\text{curl } \mathbf{F}$ using Formula (3.60)

$$\text{curl } \mathbf{F} = \frac{1}{r} \begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & \mathbf{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ F_r & rF_\theta & F_z \end{vmatrix} = \frac{1}{r} \begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & \mathbf{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ 0 & 1 & 0 \end{vmatrix}$$

$$= \mathbf{0}$$

| P. 170 # 11

Compute the gradient, in spherical coordinates, of $f(r, \phi, \theta) = \frac{\cos \phi}{r^2}$

Use formula (3.66) for gradient in spherical coords.

$$\text{grad } f(r, \phi, \theta) = \frac{\partial f}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial f}{\partial \phi} \mathbf{e}_\phi + \frac{1}{r \sin \phi} \frac{\partial f}{\partial \theta} \mathbf{e}_\theta$$

$$\frac{\partial f}{\partial r} = -\frac{2 \cos \phi}{r^3} \quad \frac{\partial f}{\partial \phi} = -\frac{\sin \phi}{r^2} \quad \frac{\partial f}{\partial \theta} = 0$$

$$\therefore \text{grad } f(r, \phi, \theta) = -\frac{2 \cos \phi}{r^3} \mathbf{e}_r - \frac{\sin \phi}{r^2} \mathbf{e}_\phi$$