

Section 4.4

P. 212 #1

$$(b) \quad \vec{F} = 2e^{xz}\vec{j} + xe^{xz}\vec{k}$$

\vec{F} is defined on \mathbb{R}^3 which is simply connected

$$\text{curl } \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 2e^{xz} & xe^{xz} \end{vmatrix} = \left(0 - \frac{\partial}{\partial z}(2e^{xz})\right)i - \left(\frac{\partial}{\partial x}(xe^{xz})\right)\vec{j} + \left(\frac{\partial}{\partial x}(2e^{xz})\right)\vec{k}$$

$$\text{curl } \vec{F} = \left(-e^{xz} - zxe^{xz}\right)\vec{i} - \left(e^{xz} + xze^{xz}\right)\vec{j} + \left(z^2 e^{xz}\right)\vec{k}$$

$$= e^{xz} \left[(-1 - zx)\vec{i} - (1 + xz)\vec{j} + z^2 \vec{k}\right] \neq 0$$

Hence \vec{F} is not conservative.

$$(1) \quad \vec{F} = 3x^2yz^2\vec{i} + x^3z^2\vec{j} + x^3y^2\vec{k}$$

\vec{F} is defined on \mathbb{R}^3 which is simply connected

\vec{F} is defined on \mathbb{R}^3

$$\text{curl } \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2yz^2 & x^3z^2 & x^3y^2 \end{vmatrix} = \left(\frac{\partial(x^3y^2)}{\partial y} - \frac{\partial(x^3z^2)}{\partial z}\right)\vec{i}$$

$$- \left(\frac{\partial(x^3z^2)}{\partial x} - \frac{\partial(3x^2yz^2)}{\partial z}\right)\vec{j} + \left(\frac{\partial(x^3z^2)}{\partial x} - \frac{\partial(3x^2y^2)}{\partial y}\right)\vec{k}$$

$$= (x^3z^2 - 2x^3z)\vec{i} - (3yzx^2 - 6y^2x^2)\vec{j} + (3x^2z^2 - 3x^2y^2)\vec{k}$$

$$= -x^3z\vec{i} + 3yzx^2\vec{j} \neq 0$$

Hence \vec{F} is not conservative.

$$(c) \quad \vec{F} = \frac{2x}{x^2+y^2}\vec{i} + \frac{2y}{x^2+y^2}\vec{j} + 2z\vec{k}$$

\vec{F} is defined on $\mathbb{R}^3 \setminus z\text{-axis}$ which is not a simply connected domain.

p12 # 1(e)(cont)

However,

$$\text{curl } \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{2x}{x^2+y^2} & \frac{2y}{x^2+y^2} & 2z \end{vmatrix} = \left(0 - \frac{2y(2x)}{x^2+y^2} \right) i - \left(\frac{\partial 2z}{\partial x} - \frac{\partial (2x)}{\partial z} \right) j + \left(\frac{\partial}{\partial x} \left[\frac{2y}{x^2+y^2} \right] - \frac{\partial (2x)}{\partial y} \right) k \\ = \left[\frac{-2y(2x)}{(x^2+y^2)^2} - \frac{(2x)(-2y)}{(x^2+y^2)^2} \right] k = 0$$

The field \vec{F} will be conservative in simply connected domains Ω that do not intersect \vec{z} -axis. But ~~it is not~~ we cannot is conservative in its full domain of definition which is not simply connected. For example on the domain (x, y, z) st $x > 0, y > 0, z > 0$ \vec{F} is conservative by the curl test.

if we can find a potential ϕ continuously differentiable on $\mathbb{R}^3 \setminus (\text{z-axis})$ then \vec{F} would be conservative on its domain of definition.

$$\frac{\partial \phi}{\partial x} = \frac{2x}{x^2+y^2} \rightarrow \phi = \int \frac{2x}{x^2+y^2} dx = \int \frac{du}{u} = \ln u + C_1(y, z) \quad \begin{matrix} \text{subst.} \\ u = x^2+y^2 \\ du = 2x dx \end{matrix}$$

$$\frac{\partial \phi}{\partial y} = \frac{2y}{x^2+y^2} \rightarrow \phi = \int \frac{2y}{x^2+y^2} dy = \ln(x^2+y^2) + C_2(x, z)$$

$$\frac{\partial \phi}{\partial z} = 2z \rightarrow \phi = \int 2z dz = z^2 + C_3(x, y)$$

$\phi = z^2 + \ln(x^2+y^2)$ is defined and continuously differentiable on $\mathbb{R}^3 \setminus z\text{-axis}$ [but is not defined for $t=0$]. $F = \nabla \phi$ $\therefore F$ is conservative !!

Section 4.4

p. 212 #3

Show that the scalar field $\phi = \frac{-1}{|\vec{R}|}$,

which is defined everywhere except at the origin, is a potential function for the vector field $\vec{F} = \frac{\vec{R}}{|\vec{R}|^3}$, where $\vec{R} = xi + yj + zk$

$$(a) \phi(r) = \frac{-1}{\sqrt{x^2+y^2+z^2}} = -(x^2+y^2+z^2)^{-1/2}$$

$$\frac{\partial \phi}{\partial x} = \left(-\frac{1}{2}\right) \frac{-2x}{(x^2+y^2+z^2)^{3/2}} = \frac{x}{(\sqrt{x^2+y^2+z^2})^3} = \frac{x}{|\vec{R}|^3}$$

$$\frac{\partial \phi}{\partial y} = \left(-\frac{1}{2}\right) \frac{-2y}{(x^2+y^2+z^2)^{3/2}} = \frac{y}{(\sqrt{x^2+y^2+z^2})^3} = \frac{y}{|\vec{R}|^3}$$

$$\frac{\partial \phi}{\partial z} = \left(-\frac{1}{2}\right) \frac{-2z}{(x^2+y^2+z^2)^{3/2}} = \frac{z}{(\sqrt{x^2+y^2+z^2})^3} = \frac{z}{|\vec{R}|^3}$$

$$\therefore \nabla \phi = \frac{(x, y, z)}{|\vec{R}|^3} = \frac{\vec{R}}{|\vec{R}|^3} = \vec{F}.$$

(b) Properties 2, 3 in Sec. 3.1 say that $\text{grad } \phi = \nabla \phi$ points in the direction of maximum rate of increase of the function ϕ , and that its magnitude equals the maximum rate of increase. The scalar field ϕ is a radial function, with negative values the reciprocal of the distance to the origin. This means the closer to the origin the more negative the function (hence the smaller). Direction of maximum rate of increase is outwards radially.

p. 212 #3 (cont)

$$\nabla \phi \parallel \vec{R}$$

The rate of maximal increase would be the same along each semiray starting at the origin

$$0 < r \quad S(r) = \frac{1}{r} \rightarrow f'(r) = -\frac{1}{r^2}$$

$$\text{so } |\nabla \phi| = \frac{1}{|R|^2}$$

$$\vec{v} \text{ unit vector } \parallel \vec{R} \text{, then } \vec{v} = \frac{\vec{R}}{|\vec{R}|}$$

$$\therefore \nabla \phi = -\frac{1}{|R|^2} \cdot \frac{\vec{R}}{|\vec{R}|} = \frac{-\vec{R}}{|R|^3} = \vec{F} \checkmark.$$

p. 212 #9 (a) Find a potential ϕ for

$$F = (2xyz + z^2 - 2y^2 + 1)\vec{i} + (x^2z - 4xy)\vec{j} + (x^2y + 2xz - 2)\vec{k}$$

$$\frac{\partial \phi}{\partial x} = 2xyz + z^2 - 2y^2 + 1 \Rightarrow \phi = \int (2xyz + z^2 - 2y^2 + 1) dx \\ = \underline{x^2yz + z^2x} - \underline{2y^2x} + x + C_1(y, z)$$

$$\frac{\partial \phi}{\partial y} = x^2z - 4xy \Rightarrow \phi = \int (x^2z - 4xy) dy = \underline{x^2yz} - \underline{2xy^2} + g(x, z)$$

$$\frac{\partial \phi}{\partial z} = (x^2y + 2xz - 2) \Rightarrow \phi = \int (x^2y + 2xz - 2) dz = \underline{x^2yz} + \underline{xz^2} - \underline{2z} + h(x, y)$$

$$\boxed{\phi(x, y, z) = x^2yz + xz^2 - 2y^2x + x - 2z}$$

clearly $F = \nabla \phi$

[P. 213 #9] (b) The field $\mathbf{G} = \frac{x}{(x^2+z^2)^2} \hat{x} + \frac{z}{(x^2+z^2)^2} \hat{z}$

satisfies the condition that $\nabla \times \mathbf{G} = \text{curl } \mathbf{G} = \mathbf{0}$
 at all points except at the y-axis where \mathbf{G}
 is not defined. Is \mathbf{G} conservative?

As in exercise #1(e) we have to try to find
 a potential to verify that \mathbf{G} is conservative,
 or show that there is no potential!

$$\frac{\partial \phi}{\partial x} = \frac{x}{(x^2+z^2)^2} \Rightarrow \phi = \int \frac{x}{(x^2+z^2)^2} dx = \frac{1}{2} \int \frac{du}{u^2} = -\frac{1}{2u} + C(y, z)$$

$u = x^2+z^2$ $du = 2x dx$

$$\frac{\partial \phi}{\partial y} = 0 \Rightarrow \phi = C(x, z)$$

$$\frac{\partial \phi}{\partial z} = \frac{z}{(x^2+z^2)^2} \Rightarrow \phi = \int \frac{z}{(x^2+z^2)^2} dz = -\frac{1}{2(x^2+z^2)} + C(x, y)$$

$\boxed{\phi = -\frac{1}{2(x^2+z^2)}}$ would be a potential for \mathbf{F}

$$\mathbf{G} = \nabla \phi$$

$\therefore \mathbf{G}$ is conservative