Given the vector field \( \vec{F} = (x + x^2) \hat{i} + xy \hat{j} + y^2 \hat{k} \), evaluate \( \text{div } \vec{F} \), \( \text{curl } \vec{F} \).

\[
\text{div } \vec{F} = \nabla \cdot \vec{F} = 1 + z^2 + x + y
\]

\[
\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
F_x & F_y & F_z
\end{vmatrix} = \left( \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \hat{i} - \left( \frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z} \right) \hat{j} + \left( \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \hat{k}
\]

Need \( \frac{\partial F_x}{\partial y} = 2z \)

\( \frac{\partial F_y}{\partial z} = y \)

\( \frac{\partial F_z}{\partial x} = 2 \)

\( \frac{\partial F_z}{\partial y} = 0 \)

Thus, \( \text{curl } \vec{F} = 2 \hat{i} + 2xz \hat{j} + y \hat{k} = \text{curl } \vec{F} \).

Can you find a vector field whose \( \text{curl } \vec{F} \) is \( \vec{A} = \vec{y} \times \vec{x} \)?

(a) \( \text{curl } \vec{F} = \vec{y} \times \vec{x} \)

(b) \( \frac{\partial F_x}{\partial y} = 0 \) and \( \frac{\partial F_y}{\partial x} = 0 \)

Guessing: let \( F_x = xy^2 \)

Then \( \frac{\partial F_x}{\partial z} = x \)

Choose \( F_y = z \) and \( F_z = xz \)
Sec 3.4

p132 #9 (a) Can you find a vector field whose curl is $y\vec{i}$?

(b) Can

(a) Looking for $F = (F_1, F_2, F_3)$ st $\text{curl} F = y\vec{i}$

If such field exists then

1. $\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z} + y$
2. $\frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z}$
3. $\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$

We will try to guess one component and use 1 2 3

Set $F_1 = xy\vec{z}$

The other

Similarly $F_2$, $F_3$ and use 1 to try to get our hands on $C_1, C_2$

Differentiate $F_2, F_3$ and use 1 to try to get our hands on $C_1, C_2$

\[
\begin{align*}
\frac{\partial F_2}{\partial z} &= \frac{x^2}{2} + \frac{2}{2} \frac{\partial}{\partial z} C_1(y, z) \\
\frac{\partial F_2}{\partial y} &= \frac{x^2}{2} + \frac{2}{2} \frac{\partial}{\partial y} C_2(y, z)
\end{align*}
\]

Choose $C_2(y, z) = \frac{y^2}{2}$, $C_1(y, z) = 0$

\[
\begin{align*}
\text{Claim} & \quad F = (xy^2)\vec{i} + \left(\frac{x^2z}{2}\right)\vec{j} + \left(\frac{xy}{2} + \frac{y^2}{2}\right)\vec{k} \\
\text{has} & \quad \text{curl} F = y\vec{i} \\
\text{Pf:} & \quad \nabla \times F = \left(\begin{array}{ccc}
\frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \frac{\partial}{\partial x} \\
xy^2 & \frac{x^2}{2} & \frac{x^2}{2} \\
\frac{x^2y}{2} & \frac{x^2}{2} & \frac{x^2}{2} + \frac{y^2}{2}
\end{array}\right) = \left(\begin{array}{ccc}
0 & 1 & 0 \\
z & 0 & 1 \\
-xy & 0 & 0
\end{array}\right) = y\vec{i}
\end{align*}
\]
(b) Can we repeat assuming this time \( \text{curl} \mathbf{F} = x \mathbf{e}_x \)?

This time

1. \( \frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z} + y \)
2. \( \frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z} \)
3. \( \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y} \)

Same guess \( F_1 = xyz \) since \( 2 \) and \( 3 \) are like in part (a)

\[ F_2 = \frac{x^2 y}{2} + C_1(y, z) \]
\[ F_3 = \frac{x^2 y}{2} + C_2(y, z) \]

Differentiating and using \( 1' \) we get this time

\[ \frac{\partial}{\partial y} \left( \frac{C_2(y, z)}{y} \right) - \frac{\partial}{\partial z} \left( \frac{C_1(y, z)}{z} \right) = x \]

But this time \( \frac{\partial C_2}{\partial z} \neq \neq \neq \frac{\partial C_1}{\partial y} \) so \( \frac{\partial C_2}{\partial z} \neq \neq \frac{\partial C_1}{\partial y} \)

but now we have a problem because

\( C_2, C_1 \) are functions of only \( y \) and \( z \).

Regardless of who \( F_1 \) is we will encounter this problem, \( 2 \) and \( 3 \) imply

\[ F_3 = \int \frac{\partial F_1}{\partial z} \, dz + C(y, z) \]
\[ F_2 = \int \frac{\partial F_1}{\partial y} \, dy + C_2(y, z) \]

We get

\[ \frac{\partial F_3}{\partial y} = \int \frac{\partial F_1}{\partial z} \, dz + \frac{\partial C_1}{\partial y} \]
\[ \frac{\partial F_2}{\partial z} = \int \frac{\partial F_1}{\partial y} \, dy + \frac{\partial C_2}{\partial z} \]

Differentiating with respect to \( y \) and \( z \) to apply \( 1' \)

\[ \text{most hold in general} \]
Math 311 - Homework #13 - Solutions - Fall 2005

We are using two facts that only are true if we assume \( F \) has continuous second-order partial derivatives:

1. \( \frac{2}{y} \int \frac{dF}{d\bar{z}} \, dx = \int \frac{\partial^2 F}{\partial y \partial z} \, dx \)

   derivative can be interchanged with integral

   \( \frac{2}{z} \int \frac{dF}{d\bar{y}} \, dx = \int \frac{\partial^2 F}{\partial z \partial y} \, dx \)

2. Mixed partial derivatives are equal

   \( \frac{\partial^2 F}{\partial z \partial y} = \frac{\partial^2 F}{\partial y \partial z} \)

Sec 3.6  
Find \( \nabla^2 f \) given \( f(x,y,z) = \frac{1}{x^2 + y^2 + z^2} \)

\( \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \)

Remember \( \frac{\partial}{\partial x} \frac{1}{R} = \frac{1}{R^3} \frac{\partial R}{\partial x} \)

\( \frac{\partial f}{\partial x} = -2x \frac{1}{R^3} \)

\( \frac{\partial f}{\partial y} = -2y \frac{1}{R^3} \)

\( \frac{\partial f}{\partial z} = -2z \frac{1}{R^3} \)

\( \Rightarrow \nabla^2 f = \Delta f = 3 \frac{(x^2 + y^2 + z^2) - 3R^2}{R^5} = 0 = \nabla^2 f \)

(f is a harmonic function)
Sec 3.6
p. 140 # 4

Which of the following functions satisfies Laplace's eqn?

(a) \( f(x, y, z) = e^z \sin y \)

\[
\frac{\partial^2 f}{\partial x^2} = 0 \\
\frac{\partial^2 f}{\partial y^2} = e^z \sin y \\
\frac{\partial^2 f}{\partial z^2} = -e^z \\
\Delta f = \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = e^z \sin y + e^z \sin y = 0
\]

(b) \( f(x, y, z) = \sin x (\sinh y) + \cos x (\cosh z) \)

\[
\frac{\partial^2 f}{\partial x^2} = -\sin x (\sinh y) \\
\frac{\partial^2 f}{\partial y^2} = \sin x (\sinh y) \\
\frac{\partial^2 f}{\partial z^2} = \cos x (\cosh z) \\
\Delta f = \nabla^2 f = 0
\]

(c) \( f(x, y, z) = \sin(px) \sinh(qy) \)

\[
\frac{\partial^2 f}{\partial x^2} = 0 \\
\frac{\partial^2 f}{\partial y^2} = -p^2 \sin(px) \sinh(qy) \\
\frac{\partial^2 f}{\partial z^2} = q^2 \sin(px) \sinh(qy) \\
\Delta f = \nabla^2 f = (q^2 - p^2) \sin(px) \sinh(qy) = 0 \iff q = \mp p
\]

f(x, y, z) = \sin(px) \sinh(qy) satisfies \( \Delta f = 0 \iff q = \mp p \)

p. 140 # 5

(a) \( \nabla f \) vector field
(b) \( \nabla \cdot F \) scalar field
(c) \( \nabla \times F \) vector field
(d) \( \nabla \cdot (\nabla f) \) scalar field
(e) \( \nabla \times (\nabla f) = 0 \) vector
(f) \( \nabla \times F \) vector field
(g) \( \nabla^2 F = \nabla \cdot (\nabla F) \) vector field
(h) \( \nabla \times (\nabla^2 f) \) non-sense
(i) \( \nabla \times (\nabla^2 f) \) scalar field
(j) \( \nabla (\nabla^2 f) \) vector field