Sec 3.4

p.132 #4 Given the vector field

$$\vec{F} = (x + xz^2)\vec{i} + xy\,\vec{j} + yz\,\vec{k}$$

evaluate $\operatorname{div}\vec{F}$, $\operatorname{curl}\vec{F}$.

Need $\frac{\partial F_1}{\partial x} = 1 + z^2$ $\quad \frac{\partial F_2}{\partial y} = x$ $\quad \frac{\partial F_3}{\partial z} = y$

$$\boxed{\operatorname{div}\vec{F} = \nabla \cdot \vec{F} = 1 + z^2 + x + y}$$

$$\operatorname{curl}\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\vec{i} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\vec{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k}$$

Need $\frac{\partial F_1}{\partial y} = 0$, $\frac{\partial F_1}{\partial z} = 2zx$; $\frac{\partial F_2}{\partial x} = y$, $\frac{\partial F_2}{\partial z} = 0$; $\frac{\partial F_3}{\partial x} = 0$, $\frac{\partial F_3}{\partial y} = z$

$$= (z - 0)\vec{i} - (0 - 2zx)\vec{j} + (y - 0)\vec{k}$$

$$= \boxed{z\vec{i} + 2zx\,\vec{j} + y\vec{k} = \operatorname{curl}\vec{F}}$$

p 132 #9 ~~Given $\vec{F} = yz\vec{i} + xz\vec{j} + xy\vec{k}$ find~~

(a) Can you find a vector field whose curl is $y\vec{i}$? $x\vec{i}$?

(a) $\operatorname{curl}\vec{F} = y\vec{i} \Rightarrow \frac{\partial F_3}{\partial x} \overset{①}{=} \frac{\partial F_1}{\partial z}$ and $\frac{\partial F_2}{\partial x} \overset{②}{=} \frac{\partial F_1}{\partial y}$

and $\frac{\partial F_3}{\partial y} \overset{③}{=} y + \frac{\partial F_2}{\partial z}$

Guessing: let $\boxed{F_1 = xyz}$ Then $\frac{\partial F_1}{\partial z} = xy \overset{①}{=} \frac{\partial F_3}{\partial x}$ $\Rightarrow F_3 = \frac{x^2 y}{2} + C(y,z)$

$\frac{\partial F_1}{\partial y} = xz \overset{②}{=} \frac{\partial F_2}{\partial x}$ $\Rightarrow F_2 = \frac{x^2 z}{2} + C_2(y,z)$

$\Rightarrow$ $\frac{\partial F_3}{\partial y} = \frac{x^2}{2} + \frac{\partial}{\partial y}C_1(y,z)$ and $\frac{\partial F_2}{\partial z} = \frac{x^2}{2} + \frac{\partial}{\partial z}C_2(y,z)$

③ $\Rightarrow \frac{\partial}{\partial y}C_1(y,z) - \frac{\partial}{\partial z}C_2(y,z) = y$

choose $C_1(y,z) = \frac{y^2}{2}$ $\quad C_2 = 0$

$\Rightarrow \boxed{F_2 = \frac{x^2 z}{2}}$ $\quad \boxed{F_1 = \frac{x^2 y}{2} + \frac{y^2}{2}}$

Sec 3.4

p132 #9 (a) Can you find a vector field whose curl is $y\vec{\imath}$?

(b) Can " " " " " " $x\vec{\imath}$?

(a) Looking for $F = (F_1, F_2, F_3)$ st $\operatorname{curl}\vec{F} = y\vec{\imath}$

If such field exists then

(1) $\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z} + y$ (2) $\frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z}$ (3) $\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$

We will try to guess one component and use (1) (2) (3) to get the others.

Set $\boxed{F_1 = xyz}$

Then $\frac{\partial F_1}{\partial y} = xz \overset{\text{by (3)}}{=} \frac{\partial F_2}{\partial y}$

$\frac{\partial F_1}{\partial z} = xy = \frac{\partial F_3}{\partial x}$

} integrating dx $\Rightarrow$ $\boxed{F_2 = \frac{x^2 z}{2} + C_1(y,z)}$ $\boxed{F_3 = \frac{x^2 y}{2} + C_2(y,z)}$

Differentiate F_2, F_3 and use (1) to try to get our hands on C_1, C_2

$\frac{\partial F_2}{\partial z} = \frac{x^2}{2} + \frac{\partial}{\partial z} C_1(y,z)$

$\frac{\partial F_3}{\partial y} = \frac{x^2}{2} + \frac{\partial}{\partial y} C_2(y,z)$

} (1) $\Rightarrow -\frac{\partial}{\partial z} C_1(y,z) + \frac{\partial}{\partial y} C_2(y,z) = y$

Choose $\boxed{C_2(y,z) = \frac{y^2}{2}}$ so that $\frac{\partial}{\partial y} C_2(y,z) = y$

$\boxed{C_1(y,z) = 0}$ so $\frac{\partial}{\partial z} C_1 = 0$

Claim $\boxed{\vec{F} = (xyz)\vec{\imath} + \left(\frac{x^2 z}{2}\right)\vec{\jmath} + \left(\frac{x^2 y}{2} + \frac{y^2}{2}\right)\vec{k}}$

has $\operatorname{curl}\vec{F} = y\vec{\imath}$

Pf: $\nabla \times \vec{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & \frac{x^2 z}{2} & \frac{x^2}{2} + \frac{y^2}{2} \end{vmatrix} = \left(\frac{x^2}{2} + y - \frac{x^2}{2}\right)\vec{\imath} - (xy - xy)\vec{\jmath} + (xz - xz)\vec{k}$

$= y\vec{\imath}$ ✓

P 132 #9 (b) Can we repeat assuming this time $\text{curl}\,\vec{F} = x\vec{i}$?

This time

(1') $\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z} + x$ (2) $\frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z}$ (3) $\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$

Same guess $F_1 = xyz$ since (2) & (3) are like in part (a)

$$F_2 = \frac{x^2 z}{2} + C_1(y,z)$$

$$F_3 = \frac{x^2 y}{2} + C_2(y,z)$$

Differentiating and using (1') we get this time

$$\boxed{\frac{\partial}{\partial y} C_2(y,z) - \frac{\partial}{\partial z} C_1(y,z) = x} \quad (*)$$

~~This time $C_2(y,z) = xy$ so $\frac{\partial}{\partial y} C_2 = x$~~

but now we have a problem because C_2, C_1 are functions of only y and z.

Regardless of who F_1 is we will encounter this problem, (2) & (3) imply

$$F_3 = \int \frac{\partial F_1}{\partial z}\,dx + G(y,z)$$

$$F_2 = \int \frac{\partial F_1}{\partial y}\,dy + C_2(y,z)$$

~~Differentiating~~ Differentiating with respect to y & z to apply (1')

we get $\frac{\partial F_3}{\partial y} = \int \frac{\partial^2 F_1}{\partial y \partial z}\,dx + \frac{\partial C_1(y,z)}{\partial y}$

$\frac{\partial F_2}{\partial z} = \int \frac{\partial^2 F_1}{\partial z \partial y}\,dx + \frac{\partial C_2(y,z)}{\partial z}$

equal

} (*) must hold in general.

MATH 311 – Homework #13 – Solutions – Fall 2005

We are using two facts that only are true if we assume $\vec{F}$ has continuous second order partial derivatives:

① $\frac{\partial}{\partial y}\int \frac{\partial F_1}{\partial z}\,dx = \int \frac{\partial^2 F_1}{\partial y \partial z}\,dx$

derivative can be interchanged with integral

$\frac{\partial}{\partial z}\int \frac{\partial F_1}{\partial y}\,dx = \int \frac{\partial^2 F_1}{\partial z \partial y}\,dx$

② Mixed partial derivatives are equal

$\frac{\partial^2 F_1}{\partial z \partial y} = \frac{\partial^2 F_1}{\partial y \partial z}$

Sec. 3.6

p. 140 #2 Find $\nabla^2 f$ given $f(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}} = \frac{1}{|R|}$

$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

Remember $\frac{\partial}{\partial x}|R| = \frac{x}{|R|}$ etc

$\frac{\partial f}{\partial x} = \frac{-\frac{\partial}{\partial x}|R|}{|R|^2} = \frac{-x}{|R|^3} \Rightarrow \frac{\partial^2 f}{\partial x^2} = \frac{-|R|^3 + x\,3|R|^2\frac{d|R|}{dx}}{|R|^6} = \frac{-|R|^3 + 3x^2|R|}{|R|^6}$

$\frac{\partial f}{\partial y} = \frac{-y}{|R|^3}$

$\frac{\partial^2 f}{\partial x^2} = \frac{3x^2 - |R|^2}{|R|^5}$ $\quad \frac{\partial^2 f}{\partial y^2} = \frac{3y^2 - |R|^2}{|R|^5}$

$\frac{\partial f}{\partial z} = \frac{-z}{|R|^3}$

$\frac{\partial^2 f}{\partial z^2} = \frac{3z^2 - |R|^2}{|R|^5}$

$\Rightarrow \nabla^2 f = \Delta f = \frac{3(x^2+y^2+z^2) - 3|R|^2}{|R|^5} = \boxed{0 = \nabla^2 f}$

f is a Harmonic function

Sec 3.6

p.140 #4 Which of the following functions satisfies Laplace's eqn?

(a) $f(x,y,z) = e^z \sin y$ **YES**

$\frac{\partial^2 f}{\partial x^2} = 0 \quad \frac{\partial^2 f}{\partial z^2} = e^z \sin y \quad \frac{\partial^2 f}{\partial y^2} = -e^z \sin y$

$\Delta f = \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = -e^z \sin y + e^z \sin y = 0$

(b) $f(x,y,z) = \sin x (\sinh y) + \cos x (\cosh z)$ **Yes**

$\frac{\partial^2 f}{\partial x^2} = -\sin x (\sinh y) - \cos x (\cosh z)$

$\frac{\partial^2 f}{\partial y^2} = \sin x (\sinh y)$

$\frac{\partial^2 f}{\partial z^2} = \cos x (\cosh z)$

$\Delta f = \nabla^2 f = 0$

$\frac{d}{dy} \sinh y = \cosh y \quad \frac{d}{dy} \cosh y = \sinh y$

$\cosh y = \frac{e^y + e^{-y}}{2} \quad \sinh y = \frac{e^y - e^{-y}}{2}$

(c) $f(x,y,z) = \sin(px) \sinh(qy)$ (p, q constants)

$\frac{\partial^2 f}{\partial z^2} = 0 \quad \frac{\partial^2 f}{\partial x^2} = -p^2 \sin(px) \sinh(qy) \quad \frac{\partial^2 f}{\partial y^2} = q^2 \sin(px) \sinh(qy)$

$\Delta f = \nabla^2 f = (q^2 - p^2) \sin(px) \sinh(qy) = 0 \iff q = \pm p$

$f(x,y,z) = \sin(px) \sinh(qy)$ satisfies $\Delta f = 0 \iff q = \pm p$

p140 #5

(a) ∇f vector field

(b) $\nabla \cdot F$ scalar field

(c) $\nabla \times F$ vector field

(d) $\nabla \cdot (\nabla f)$ scalar field

(e) $\nabla \times (\nabla f) = \vec{0}$ vector $\vec{0}$

(f) $\nabla \times f$ makes no sense

(g) $\nabla^2 F$ = [illegible] makes sense if we define it as acting on components

$\nabla^2 F = (\nabla^2 F_1, \nabla^2 F_2, \nabla^2 F_3)$ vector field

(h) $\nabla \times (\nabla^2 F)$ vector field

(i) $\nabla \times (\nabla^2 f)$ non-sense ($\nabla^2 f$ scalar)

(j) $\nabla (\nabla^2 f)$ vector field