

## Section 3.3

p. 124 #4

Find the divergence of the field

$$\mathbf{F}(x, y, z) = \frac{x\vec{i} + y\vec{j} + z\vec{k}}{(\sqrt{x^2 + y^2 + z^2})^3}$$

defined  
for all  
(x, y, z) ≠ (0, 0, 0)denote by  $\vec{R}$  the identity vector field

$$\mathbf{R}(x, y, z) = x\vec{i} + y\vec{j} + z\vec{k} \quad \text{Then } \mathbf{F} = \frac{\mathbf{R}}{|\mathbf{R}|^3}$$

scalar field

$$|\mathbf{R}(x, y, z)| = \sqrt{x^2 + y^2 + z^2}$$

$$\text{Notice that } \frac{d}{dx} |\mathbf{R}|^3 = 3|\mathbf{R}|^2 \cdot \frac{\partial}{\partial x} |\mathbf{R}| = 3|\mathbf{R}|^2 \cdot \frac{\partial}{\partial x} |R|$$

$$\frac{\partial}{\partial x} |\mathbf{R}|^3 = 3 \times |\mathbf{R}| \quad \frac{\partial}{\partial y} |\mathbf{R}|^3 = 3y |\mathbf{R}| \quad \frac{\partial}{\partial z} |\mathbf{R}|^3 = 3z |\mathbf{R}|$$

$$\text{Similarly } \frac{\partial}{\partial y} |\mathbf{R}|^3 = 3y |\mathbf{R}| \quad \frac{\partial}{\partial z} |\mathbf{R}|^3 = 3z |\mathbf{R}|$$

To compute divergence we need  $\frac{\partial F_1}{\partial x}, \frac{\partial F_2}{\partial y}, \frac{\partial F_3}{\partial z}$ 

$$\text{where } F_1 = \frac{x}{|\mathbf{R}|^3}, \quad F_2 = \frac{y}{|\mathbf{R}|^3}, \quad F_3 = \frac{z}{|\mathbf{R}|^3}$$

$$\text{Now } \frac{\partial F_1}{\partial x} = \frac{1 \cdot |\mathbf{R}|^3 - x \frac{\partial |\mathbf{R}|^3}{\partial x}}{|\mathbf{R}|^6} = \frac{|\mathbf{R}|^3 - 3x^2 |\mathbf{R}|}{|\mathbf{R}|^6} = \frac{|\mathbf{R}|^2 - 3x^2}{|\mathbf{R}|^5}$$

$$\text{similarly } \frac{\partial F_2}{\partial y} = \frac{|\mathbf{R}|^2 - 3y^2}{|\mathbf{R}|^5}, \quad \frac{\partial F_3}{\partial z} = \frac{|\mathbf{R}|^2 - 3z^2}{|\mathbf{R}|^5}$$

$$\text{Finally: } \operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= \frac{3|\mathbf{R}|^2 - 3(x^2 + y^2 + z^2)}{|\mathbf{R}|^5} = 0$$

ANS

$\operatorname{div} \mathbf{F} = 0$	for all $(x, y, z) \neq (0, 0, 0)$
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## Section 3.3

p.124 #6

Find non-constant scalar and vector fields

$$\phi: \mathbb{R}^3 \rightarrow \mathbb{R}, \quad \vec{F}: \mathbb{R}^3 \rightarrow \mathbb{R}^3$$

such that  $\boxed{\operatorname{div}(\phi \vec{F}) = \phi \operatorname{div} F} \quad (*)$

First note that  $*$  is in general not true:

$$\operatorname{div}(\phi \vec{F}) = \frac{\partial(\phi F_1)}{\partial x} + \frac{\partial(\phi F_2)}{\partial y} + \frac{\partial(\phi F_3)}{\partial z}$$

by product rule  $\leftarrow = \left( \phi \frac{\partial F_1}{\partial x} + \frac{\partial \phi}{\partial x} F_1 \right) + \left( \phi \frac{\partial F_2}{\partial y} + \frac{\partial \phi}{\partial y} F_2 \right) + \left( \phi \frac{\partial F_3}{\partial z} + \frac{\partial \phi}{\partial z} F_3 \right)$

$$= \phi \left( \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) + \nabla \phi \cdot \vec{F}$$

$$\therefore \operatorname{div}(\phi \vec{F}) = \phi \operatorname{div} F + \underbrace{\nabla \phi \cdot \vec{F}}$$

For equality to hold in  $*$  we need that  $\boxed{\nabla \phi \cdot \vec{F} = 0}$ 

Answer

For example consider:

$$F(x, y, z) = \vec{x}$$

$$\phi(x, y, z) = g(y, z) \quad (\text{so that } \frac{\partial \phi}{\partial x} = 0)$$

$$\text{Then } \nabla \phi = \left( 0, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} \right) \text{ and}$$

$$\nabla \phi \cdot F = \left( 0, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} \right) \cdot (x, 0, 0) = 0$$

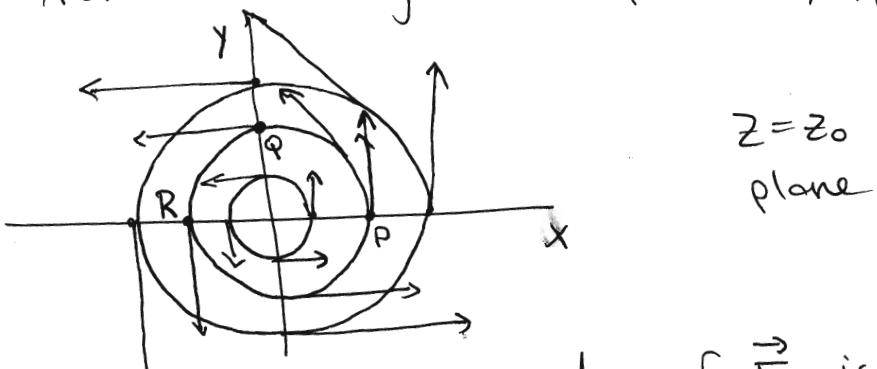
where  $g$  is your favorite non-constant two variable function.

Section 3.3

p.124 #11

The vector

What can you say about the divergence of the field in the figure at points P, Q, R?

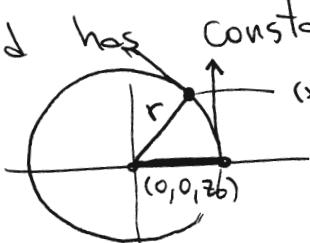


Assume the variation in the z-direction of  $\vec{F}$  is zero, and that  $F_3 \equiv 0$ . This means:

$$\vec{F}(\vec{x}) = F_1(x, y) \vec{i} + F_2(x, y) \vec{j} \quad \text{at } \vec{x} = (x, y, z)$$

Therefore  $\nabla \cdot \vec{F} = \operatorname{div} \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}$

The picture seems to indicate that the field is tangent to circles centered at  $(0, 0, z_0)$  on the  $z=z_0$  plane and has constant magnitude the radius of the circle



That is  $r = \sqrt{x^2 + y^2}$

$$\vec{F} = -y \vec{i} + x \vec{j} \quad (*)$$

$$|\vec{F}| = \sqrt{x^2 + y^2} = r$$

Now we can compute if (\*) is indeed the right formula:

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} = 0 + 0 = 0$$

This is an example of a divergence free vector field

Heuristically



"arrows coming in are balanced by arrows coming out".