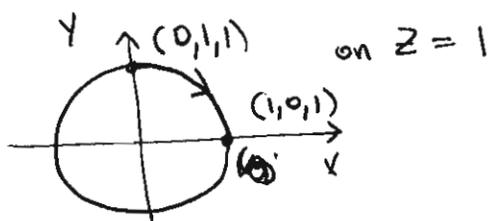


1. $\vec{F} = (yz + 2x)\vec{i} + xz\vec{j} + (xy + 2z)\vec{k}$

$C: x^2 + y^2 = 1 \quad z = 1$ from $(0, 1, 1)$ to $(1, 0, 1)$



Parametrization: $0 \leq \theta \leq \frac{\pi}{2}$

$$\begin{cases} x = \sin \theta & \theta = 0 \rightarrow (0, 1, 1) \\ y = \cos \theta & \theta = \frac{\pi}{2} \rightarrow (1, 0, 1) \\ z = 1 \end{cases}$$

$\int_C \vec{F} \cdot d\vec{R} = ?$

$d\vec{R} = (\cos \theta, -\sin \theta, 0)$

$F(R(\theta)) = (\cos^2 \theta + 2 \sin \theta)\vec{i} + \sin \theta \vec{j} + (\sin \theta \cos \theta + 2)\vec{k}$

$\int_C \vec{F} \cdot d\vec{R} = \int_0^{\pi/2} [\cos^2 \theta + 2 \sin \theta \cos \theta - \sin^2 \theta] d\theta$

Trig identities

$\sin(2\theta) = 2 \sin \theta \cos \theta$
 $\cos(2\theta) = \cos^2 \theta - \sin^2 \theta$

$= \int_0^{\pi/2} [\sin(2\theta) + \cos(2\theta)] d\theta$

$= \left[-\frac{\cos(2\theta)}{2} + \frac{\sin(2\theta)}{2} \right]_{\theta=0}^{\theta=\pi/2}$

$= \frac{-\cos \pi + \cos 0}{2} + \frac{\sin \pi - \sin 0}{2}$

$= \frac{-(-1) + 1}{2} = \boxed{1}$

2. Show that $\vec{F} = (4xy - 3x^2z^2)\vec{i} + 2x^2\vec{j} + 2x^3z\vec{k}$ is conservative. Find scalar potential ϕ .

$\frac{\partial \phi}{\partial x} = 4xy - 3x^2z^2 \xrightarrow{\text{integration}} \phi = \frac{4x^2y}{2} - x^3z^2 + C(y, z)$

$\frac{\partial \phi}{\partial y} = 2x^2 \xrightarrow{dy} \phi = 2x^2y + C(x, z)$

$\frac{\partial \phi}{\partial z} = -2x^3z \xrightarrow{dz} \phi = -x^3z^2 + C(x, y)$

2. (cont) $\phi = 2x^2y - x^3z^2$ is a potential

since $\nabla\phi = (4xy - 3x^2z^2, 2x^2, -2x^3z) = \vec{F}$

$$\int_{(0,2,-3)}^{(-1,1,2)} \vec{F} \cdot d\vec{R} = \int_{(0,2,-3)}^{(-1,1,2)} \nabla\phi \cdot d\vec{R} = \phi(-1,1,2) - \phi(0,2,-3)$$

$$= (2(-1)^2 \cdot 1 - (-1)^3(2)^2) - (0 - 0)$$

$$= 2 - (-4) = \boxed{6}$$

3. $\vec{F} = 2xz\vec{i} + (x^2 - y)\vec{j} + (2z - x^2)\vec{k}$, is \vec{F} conservative?

Suppose it is, then $\exists \phi$ s.t. $F = \nabla\phi$

$\frac{\partial\phi}{\partial x} = 2xz$ Integrating \xrightarrow{dx} $\phi = x^2z + C(y,z)$ (1)

$\frac{\partial\phi}{\partial y} = x^2 - y$ \xrightarrow{dy} $\phi = x^2y - \frac{y^2}{2} + C(x,z)$

$\frac{\partial\phi}{\partial z} = 2z - x^2$ \xrightarrow{dz} $\phi = z^2 - x^2z + C(x,y)$

These 3 identities cannot hold simultaneously. (*)

Hence \vec{F} is NOT a conservative field

(*) let me elaborate from (1) suppose $\phi = x^2z + C(y,z)$ then $\frac{\partial\phi}{\partial y} = \frac{\partial C(y,z)}{\partial y}$ a function of only y, z

however by hypothesis $\frac{\partial\phi}{\partial y} = x^2 - y$ a function of x and y .

These two statements are incompatible.

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$$4. \quad \vec{F} = \overbrace{(2x+3y)\vec{i}}^{F_1} + \overbrace{(x-2y)\vec{j}}^{F_2}$$

$$\text{div } F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} = 2 - 2 = 0$$

$$(a) \quad \kappa \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & 1 \\ 2x+3y & x-2y & 0 \end{vmatrix} = -(x-2y)\vec{i} + (2x+3y)\vec{j}$$

$$\text{curl}(\kappa \times \vec{F}) = \text{div } F \vec{k} = 0$$

$\exists \chi$ stream function (~~vector~~ ^{scalar} potential for $\kappa \times \vec{F}$)

$$\frac{\partial \chi}{\partial x} = -(x-2y) \quad \int dx \rightarrow \chi = -\frac{x^2}{2} + 2xy + C(y, z)$$

$$\frac{\partial \chi}{\partial y} = 2x+3y \quad \int dy \quad \chi = 2xy + \frac{3y^2}{2} + C(x, z)$$

$$\frac{\partial \chi}{\partial z} = 0 \quad \int dz \quad \chi = C(x, y)$$

$$\boxed{\chi(x, y, z) = -\frac{x^2}{2} + 2xy + \frac{3y^2}{2}}$$

Does the job.

$$\text{let } G_1 = \chi \vec{k} = \left(-\frac{x^2}{2} + 2xy + \frac{3y^2}{2}\right) \vec{k}$$

$$\text{Then } \nabla \times G_1 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & -\frac{x^2}{2} + 2xy + \frac{3y^2}{2} \end{vmatrix} = (2x+3y)\vec{i} - (-x+2y)\vec{j}$$

$$\nabla \times G_1 = (2x+3y)\vec{i} + (x-2y)\vec{j} = \vec{F} \quad \checkmark$$

$$(b) \quad \vec{r}(t) = t(x, y, z) \quad \frac{d\vec{r}}{dt} = (x, y, z)$$

$$F(\vec{r}(t)) = (2tx + 3ty)\vec{i} + (tx - 2ty)\vec{j}$$

$$\vec{F} \times \frac{d\vec{r}}{dt} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ t(2x+3y) & t(x-2y) & 0 \\ x & y & z \end{vmatrix} = t z (x-2y) \vec{i} - t z (2x+3y) \vec{j} + [t y (2x+3y) - t x (x-2y)] \vec{k}$$

$$\begin{aligned} G &= \int_0^1 t \vec{F} \times \frac{d\vec{r}}{dt} dt = \int_0^1 t^2 dt [z(x-2y) \vec{i} - z(2x+3y) \vec{j}] \\ &\quad + \int_0^1 t^2 dt [y(2x+3y) - x(x-2y)] \vec{k} \\ &= \frac{t^3}{3} \Big|_0^1 (z(x-2y) \vec{i} - z(2x+3y) \vec{j}) \\ &\quad + \frac{t^3}{3} \Big|_0^1 [2xy + 3y^2 - x^2 + 2xy] \vec{k} \end{aligned}$$

$$\vec{G} = \frac{(2x-2yz) \vec{i}}{3} - \frac{(2zx+3zy) \vec{j}}{3} + \frac{(4xy+3y^2-x^2) \vec{k}}{3}$$

$$\nabla \times \vec{G} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{2(x-2y)}{3} & -\frac{z(2x+3y)}{3} & \frac{4xy+3y^2-x^2}{3} \end{vmatrix} = \left[\frac{4x+6y+2x+3y}{3} \right] \vec{i} + \left[\frac{4y-2x}{3} - \frac{(x-2y)}{3} \right] \vec{j} + \left(\frac{-2z}{3} + \frac{2z}{3} \right) \vec{k}$$

$$= (2x+3y) \left(\frac{2}{3} + \frac{1}{3} \right) \vec{i} + (x-2y) \left(\frac{1}{3} + \frac{2}{3} \right) \vec{j}$$

$$= (2x+3y) \vec{i} + (x-2y) \vec{j} = \vec{F}$$

✓

(cont) 4. (c) Compare G_1, G .

$$G_1(x,y,z) = \left(\frac{-x^2 + 4xy + 3y^2}{2} \right) \vec{k}$$

$$G(x,y,z) = \left(\frac{2x - 2yz}{3} \right) \vec{i} + \left(\frac{2zx + 3zy}{3} \right) \vec{j} + \frac{2}{3} G_1(x,y,z)$$

so $G - G_1 = \left(\frac{2x - 2yz}{3} \right) \vec{i} - \left(\frac{2zx + 3zy}{3} \right) \vec{j} - \frac{1}{3} G_1(x,y,z)$

$$\frac{\partial \phi}{\partial x} = \frac{2x - 2yz}{3} \quad \int dx \rightarrow \phi = \frac{2x^2}{6} - \frac{2yzx}{3} + C(y,z)$$

$$\frac{\partial \phi}{\partial y} = -\frac{2zx + 3zy}{3} \rightarrow \phi = -\frac{2yzx}{3} - \frac{3zy^2}{6} + C(x,z)$$

$$\frac{\partial \phi}{\partial z} = \frac{x^2 - 4xy - 3y^2}{6} \rightarrow \phi = \frac{zx^2}{6} - \frac{4xy^2z}{6} - \frac{3y^2z^2}{6} + C(x,y)$$

$$\phi = \frac{zx^2}{6} - \frac{2yzx}{3} - \frac{y^2z}{2}$$

does the job

that is $G - G_1 = \nabla \phi$

5. $\vec{F} = 2y \vec{i} - z \vec{j} + 3x \vec{k}$

(a) $x = \rho \cos \theta$
 $y = \rho \sin \theta$
 $z = z$

$$\left. \begin{aligned} \vec{e}_\rho &= \cos \theta \vec{i} + \sin \theta \vec{j} \\ \vec{e}_\theta &= -\sin \theta \vec{i} + \cos \theta \vec{j} \\ \vec{e}_z &= \vec{k} \end{aligned} \right\} \begin{aligned} \vec{i} &= \cos \theta \vec{e}_\rho - \sin \theta \vec{e}_\theta \\ \vec{j} &= \sin \theta \vec{e}_\rho + \cos \theta \vec{e}_\theta \\ \vec{e}_z &= \vec{k} \end{aligned}$$

$$\vec{F} = 2\rho \sin \theta (\cos \theta \vec{e}_\rho - \sin \theta \vec{e}_\theta) - z (\sin \theta \vec{e}_\rho + \cos \theta \vec{e}_\theta) + 3\rho \cos \theta \vec{e}_z$$

$$\vec{F} = \underbrace{(2\rho \sin \theta \cos \theta - z \sin \theta)}_{F_\rho} \vec{e}_\rho - \underbrace{(2\rho \sin^2 \theta + z \cos \theta)}_{F_\theta} \vec{e}_\theta + \underbrace{3\rho \cos \theta}_{F_z} \vec{e}_z$$

5. (b) (cont) (b) $x = r \sin \phi \cos \theta$
 $y = r \sin \phi \sin \theta$
 $z = r \cos \phi$

$$\begin{aligned} \vec{e}_r &= \sin \phi \cos \theta \vec{i} + \sin \phi \sin \theta \vec{j} + \cos \phi \vec{k} \\ \vec{e}_\theta &= -\sin \theta \vec{i} + \cos \theta \vec{j} \\ \vec{e}_\phi &= +\cos \phi \cos \theta \vec{i} + \cos \phi \sin \theta \vec{j} - \sin \phi \vec{k} \end{aligned}$$

Instead of writing $\vec{i}, \vec{j}, \vec{k}$ in terms of $\vec{e}_r, \vec{e}_\theta, \vec{e}_\phi$

I will replace x, y, z in \vec{F} and hope $\vec{e}_r, \vec{e}_\theta, \vec{e}_\phi$ appear naturally:

* $\vec{F} = 2r \sin \phi \sin \theta \vec{i} - r \cos \phi \vec{j} + 3r \sin \phi \cos \theta \vec{k}$

Not obvious, we'll have to solve for $\vec{i}, \vec{j}, \vec{k}$

$$\cos \phi \vec{e}_r - \sin \phi \vec{e}_\phi = \vec{k}$$

$$\begin{aligned} \sin \phi \vec{e}_r + \cos \phi \vec{e}_\phi &= \cos \theta \vec{i} + \sin \theta \vec{j} \\ \vec{e}_\theta &= -\sin \theta \vec{i} + \cos \theta \vec{j} \end{aligned}$$

From these 2 eqns:

$$\begin{aligned} \sin \theta \sin \phi \vec{e}_r + \sin \theta \cos \phi \vec{e}_\phi + \cos \theta \vec{e}_\theta &= \vec{j} \\ \cos \theta \sin \phi \vec{e}_r + \cos \theta \cos \phi \vec{e}_\phi - \sin \theta \vec{e}_\theta &= \vec{i} \end{aligned}$$

Now can plug into *

$$\begin{aligned} \vec{F} &= 2r \sin \phi \sin \theta [\cos \theta \sin \phi \vec{e}_r + \cos \theta \cos \phi \vec{e}_\phi - \sin \theta \vec{e}_\theta] \\ &\quad - r \cos \phi [\sin \theta \sin \phi \vec{e}_r + \sin \theta \cos \phi \vec{e}_\phi + \cos \theta \vec{e}_\theta] \\ &\quad + 3r \sin \phi \cos \theta [\cos \phi \vec{e}_r - \sin \phi \vec{e}_\phi] \end{aligned}$$

$\vec{F} =$

$$\begin{aligned} &= (2r \sin^2 \phi \sin \theta \cos \theta - r \cos \phi \sin \theta \cos \theta + 3r \sin \phi \cos \theta \cos \phi) \vec{e}_r \\ &\quad + (2r \sin \phi \sin \theta \cos \phi \cos \theta - r \sin \theta \cos^2 \phi) \vec{e}_\theta \\ &\quad + (-2r \sin \phi \sin^2 \theta + r \cos \phi \cos \theta - 3r \sin^2 \phi \cos \theta) \vec{e}_\phi \end{aligned}$$

6. Heat Equation $\frac{\partial U}{\partial t} = \kappa \nabla^2 U$

in spherical coordinates: (Formula Laplace p.180)

$$\frac{\partial U}{\partial t} = \kappa \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial U}{\partial r} \right) + \frac{1}{r^2 \sin \phi} \frac{\partial}{\partial \phi} \left(\sin \phi \frac{\partial U}{\partial \phi} \right) + \frac{1}{r^2 \sin^2 \phi} \frac{\partial^2 U}{\partial \theta^2} \right]$$

(a) U is independent of $\phi \Rightarrow \frac{\partial U}{\partial \phi} = 0$

$$\frac{\partial U}{\partial t} = \kappa \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial U}{\partial r} \right) + \frac{1}{r^2 \sin^2 \phi} \frac{\partial^2 U}{\partial \theta^2} \right]$$

(b) U is ind. of ϕ and $\theta \Rightarrow \frac{\partial U}{\partial \phi} = \frac{\partial U}{\partial \theta} = 0$

$$\frac{\partial U}{\partial t} = \kappa \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial U}{\partial r} \right) \quad \leftarrow \text{(For radial functions)}$$

(c) U is ind. of r and $t \Rightarrow \frac{\partial U}{\partial t} = 0 = \frac{\partial U}{\partial r}$

$$0 = \kappa \left[\frac{1}{r^2 \sin \phi} \frac{\partial}{\partial \phi} \left(\sin \phi \frac{\partial U}{\partial \phi} \right) + \frac{1}{r^2 \sin^2 \phi} \frac{\partial^2 U}{\partial \theta^2} \right]$$

(d) U is ind of $\phi, \theta, t \Rightarrow \frac{\partial U}{\partial \phi} = \frac{\partial U}{\partial \theta} = \frac{\partial U}{\partial t} = 0$

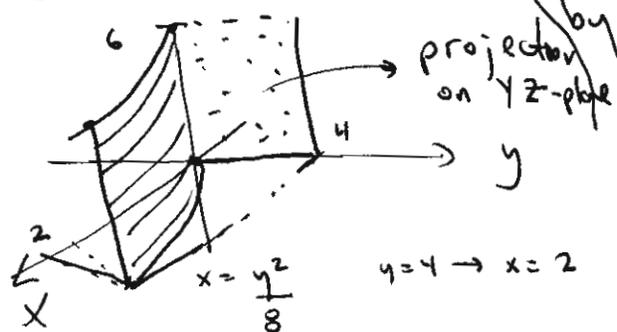
$$0 = \kappa \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial U}{\partial r} \right) \Rightarrow \frac{\partial}{\partial r} \left(r^2 \frac{\partial U}{\partial r} \right) = 0$$

$$\Rightarrow r^2 \frac{\partial U}{\partial r} = C \Rightarrow \frac{\partial U}{\partial r} = \frac{C}{r^2} \Rightarrow \boxed{U = -\frac{C}{r} + a}$$

Solution to bonus exam 1!!

7. $\vec{F} = 2y\vec{i} - z\vec{j} + x^2\vec{k}$

$S: y^2 = 8x$ first octant, bounded by planes $y=4, z=6$



Parametrize by y, z

$0 \leq y \leq 4$

$0 \leq z \leq 6$

$x = \frac{y^2}{8} \quad \frac{\partial R}{\partial y} = \left(\frac{y}{4}, 1, 0\right)$

$y = y \quad \frac{\partial R}{\partial z} = (0, 0, 1)$

$z = z$

Compute

$$\iint_S \vec{F} \cdot \vec{n} \, dS = \iint_S \vec{F} \cdot d\vec{S}$$

$$d\vec{S} = \left(\frac{\partial R}{\partial y} \times \frac{\partial R}{\partial z} \right) dy \, dz = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{y}{4} & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} dy \, dz =$$

$$= \left(\vec{i} - \frac{y}{4} \vec{j} \right) dy \, dz$$

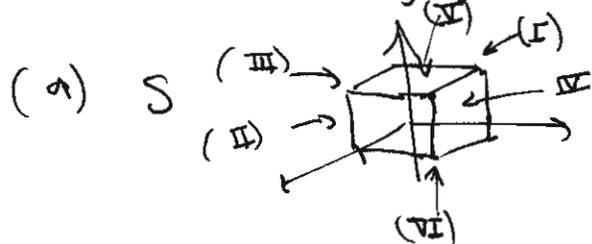
$$\iint_S \vec{F} \cdot d\vec{S} = \int_0^6 \int_0^4 \left[2y\vec{i} - z\vec{j} + \frac{y^4}{64}\vec{k} \right] \cdot \left[\vec{i} - \frac{y}{4}\vec{j} \right] dy \, dz$$

$$= \int_0^6 \int_0^4 \left(2y - \frac{yz}{4} \right) dy \, dz = \int_0^6 \left(y^2 - \frac{y^2 z}{8} \right) \Big|_{y=0}^{y=4} dz$$

$$= \int_0^6 \left(16 - \frac{16z}{8} \right) dz = (16z - z^2) \Big|_{z=0}^6$$

$$= 16 \cdot 6 - 6^2 = 96 - 36 = \boxed{60}$$

8. $\vec{F} = x\vec{i} + y\vec{j} + z\vec{k}$ find $\iint_S \vec{F} \cdot d\vec{S}$



unit cube
 $0 \leq x \leq 1$
 $0 \leq y \leq 1$
 $0 \leq z \leq 1$

6 faces
 (I) & (II) || yz plane
 (III) & (IV) || xz plane
 (V) & (VI) || yx plane

Faces (I) & (II)

(I) $x=0$ $0 \leq y, z \leq 1$ $\vec{F} = y\vec{j} + z\vec{k}$ $\vec{n} = -\vec{i}$
 $\vec{F} \cdot \vec{n} = 0 \rightarrow \iint_{(I)} \vec{F} \cdot d\vec{S} = 0$
 $\vec{n} = \vec{i}$

(II) $x=1$ $0 \leq y, z \leq 1$ $\vec{F} = \vec{i} + y\vec{j} + z\vec{k}$ $\vec{n} = \vec{i}$
 $\vec{F} \cdot \vec{n} = 1 \rightarrow \iint_{(II)} \vec{F} \cdot d\vec{S} = \text{Area}_{(II)} = 1$

(III) $y=0$ $0 \leq x, z \leq 1$ $\vec{F} = x\vec{i} + z\vec{k}$ $\vec{n} = -\vec{j}$
 $\vec{F} \cdot \vec{n} = 0 \rightarrow \iint_{(III)} \vec{F} \cdot d\vec{S} = 0$

(IV) $y=1$ $0 \leq x, z \leq 1$ $\vec{F} = x\vec{i} + \vec{j} + z\vec{k}$ $\vec{n} = \vec{j}$
 $\vec{F} \cdot \vec{n} = 1 \rightarrow \iint_{(IV)} \vec{F} \cdot d\vec{S} = \text{Area}_{(IV)} = 1$

(V) $z=1$ $0 \leq x, y \leq 1$ $\vec{F} = x\vec{i} + y\vec{j} + \vec{k}$ $\vec{n} = \vec{k}$
 $\vec{F} \cdot \vec{n} = 1 \rightarrow \iint_{(V)} \vec{F} \cdot d\vec{S} = \text{Area}_{(V)} = 1$

(VI) $z=0$ $0 \leq x, y \leq 1$ $\vec{F} = x\vec{i} + y\vec{j}$ $\vec{n} = -\vec{k}$
 $\vec{F} \cdot \vec{n} = 0 \rightarrow \iint_{(VI)} \vec{F} \cdot d\vec{S} = 0$

$\iint_S (x\vec{i} + y\vec{j} + z\vec{k}) \cdot d\vec{S} = 1 + 1 + 1 = \boxed{3}$



(cont)

8. (b) S surface of sphere of radius a . centered at the origin.

Spherical coordinates: $x = a \sin \phi \cos \theta$
 $r = a$ $0 \leq \theta \leq 2\pi$ $y = a \sin \phi \sin \theta$
 $0 \leq \phi \leq \pi$ $z = a \cos \phi$

$\vec{n} \, dS = \vec{dS} = \left(\frac{\partial R}{\partial \phi} \times \frac{\partial R}{\partial \theta} \right) d\phi \, d\theta$

$\frac{\partial R}{\partial \phi} = (a \cos \phi \cos \theta, a \cos \phi \sin \theta, -a \sin \phi)$

$\frac{\partial R}{\partial \theta} = (a \sin \phi \sin \theta, a \sin \phi \cos \theta, 0)$

$\frac{\partial R}{\partial \phi} \times \frac{\partial R}{\partial \theta} = \begin{vmatrix} a \cos \phi \cos \theta & a \cos \phi \sin \theta & -a \sin \phi \\ a \sin \phi \sin \theta & a \sin \phi \cos \theta & 0 \end{vmatrix} = \begin{matrix} -a^2 \sin^2 \phi \cos \theta \, \hat{i} \\ +a^2 \sin^2 \phi \sin \theta \, \hat{j} \\ a^2 \cos \phi \sin \phi \, \hat{k} \end{matrix}$
 (where $\hat{i}, \hat{j}, \hat{k}$ are unit vectors)

$= a^2 \sin \phi \underbrace{(\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi)}_{\substack{(x, y, z) \\ a} = \vec{n}}$

$\therefore \iint_S F \cdot \vec{dS} = \int_0^{2\pi} \int_0^{\pi} \cancel{a^2 \sin \phi \cos \theta} (x, y, z) \cdot \frac{(x, y, z)}{a} \cdot a \sin \phi \, d\phi \, d\theta$
 $= \int_0^{2\pi} \int_0^{\pi} a (x^2 + y^2 + z^2) \sin \phi \, d\phi \, d\theta$
 $= a^3 \int_0^{2\pi} \int_0^{\pi} \sin \phi \, d\phi \, d\theta = a^3 \int_0^{2\pi} [-\cos \phi]_0^{\pi} \, d\theta =$
 $= a^3 (+1 - (-1)) \int_0^{2\pi} d\theta = \boxed{4\pi a^3}$

