

1. $\vec{F} = y\vec{i} - x\vec{j} + z\vec{k}$. Evaluate $\int_C \vec{F} \cdot d\vec{R}$

from $(1,0,0)$ to $(1,0,4)$

(a) Along the segment joining $(1,0,0)$ to $(1,0,4)$
 direction of the segment = $(1,0,4) - (1,0,0) = (0,0,4)$

Parametrization $R(t) = (1,0,0) + t(0,0,4)$

$R(0) = (1,0,0)$

$R(1) = (1,0,4)$

$0 \leq t \leq 1$

$\vec{R}(t) = (1,0,t4)$, $\frac{d\vec{R}}{dt} = (0,0,4)$

$\vec{F}(\vec{R}(t)) = -\vec{j} + 4t\vec{k}$

$\vec{F}(\vec{R}(t)) \cdot \frac{d\vec{R}}{dt} = (0, -1, 4t) \cdot (0, 0, 4) = 16t$

$\int_{C_1} \vec{F} \cdot d\vec{R} = \int_0^1 16t dt = \left. \frac{16t^2}{2} \right|_0^1 = \boxed{8}$

(b) Along the helix $x = \cos t$, $y = \sin t$, $z = \frac{4t}{2\pi}$
 $0 \leq t \leq 2\pi$
 $\frac{d\vec{R}}{dt} = (-\sin t, \cos t, \frac{4}{2\pi})$

$\vec{F}(\vec{R}(t)) = \sin t \vec{i} - \cos t \vec{j} + \frac{4t}{2\pi} \vec{k}$

$\vec{F}(\vec{R}(t)) \cdot \frac{d\vec{R}}{dt} = (\sin t, -\cos t, \frac{4t}{2\pi}) \cdot (-\sin t, \cos t, \frac{4}{2\pi})$

$= -\sin^2 t - \cos^2 t + \frac{4t}{\pi^2} = -1 + \frac{4t}{\pi^2}$

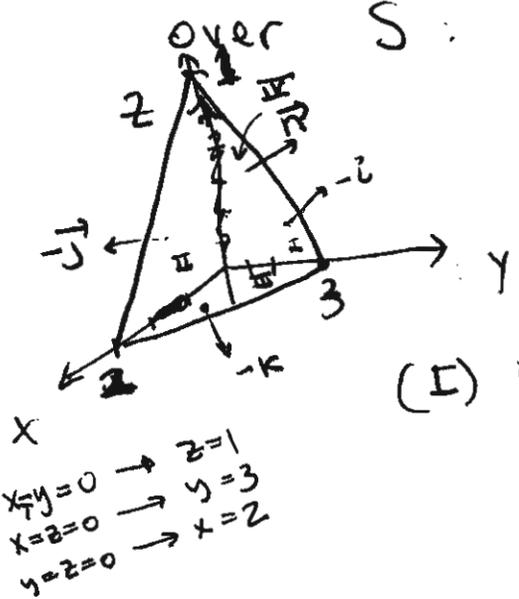
$\int_{C_2} \vec{F} \cdot d\vec{R} = \int_0^{2\pi} (-1 + \frac{4t}{\pi^2}) dt = \left. -t + \frac{2t^2}{\pi^2} \right|_0^{2\pi} = \boxed{-2\pi + 8}$

(c) \vec{F} is not conservative because we just showed that the line integrals along two different paths connecting $(1,0,0)$ & $(1,0,4)$ are different. [can also verify that $\text{curl } \vec{F} \neq 0$]

2. Evaluate $\iint_S \vec{F} \cdot \vec{n} \, dS$ for $\vec{F} = xy\vec{i} + y^2\vec{j} + zy\vec{k}$

S: Tetrahedron bounded by $x=0, y=0, z=0$ and $3x + 2y + 6z = 6$

We have to account for 4 faces



(I) On plane $x=0$, normal = $-\vec{i}$

$$\vec{F}(0, y, z) = y^2\vec{j} + zy\vec{k}$$

$$\vec{F} \cdot (-\vec{i}) = 0 \Rightarrow \iint_{(I)} \vec{F} \cdot \vec{n} \, dS = 0$$

(II) On plane $y=0$, normal = $-\vec{j}$

$$F(x, 0, z) = 0 \Rightarrow F \cdot n = 0 \Rightarrow \iint_{(II)} \vec{F} \cdot \vec{n} \, dS = 0$$

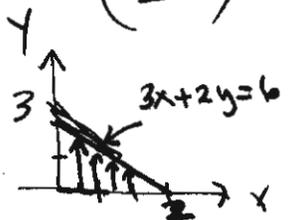
(III) On plane $z=0$, normal = $-\vec{k}$

$$F(x, y, 0) = xy\vec{i} + y^2\vec{j}$$

$$F \cdot (-\vec{k}) = 0 \Rightarrow \iint_{(III)} \vec{F} \cdot \vec{n} \, dS = 0$$

Only contribution comes from the slanted plane

(IV) Parametrize plane: $3x + 2y + 6z = 6$ $z=0$
 $y = \frac{6-3x}{2}$
 $y = 3 - \frac{3x}{2}$



$$\begin{cases} x = x \\ y = y \\ z = 1 - \frac{x}{2} - \frac{y}{3} \end{cases}$$

$$0 \leq x \leq 2$$

$$0 \leq y \leq 3 - \frac{3x}{2}$$

$$\vec{n} = \frac{(3, 2, 6)}{\sqrt{9+4+36}} = \frac{(3, 2, 6)}{7}$$

(pointing outwards)

$$\iint_{IV} \vec{F} \cdot \vec{n} \, dS = \iint_{IV} F(x, y, 1 - \frac{x}{2} - \frac{y}{3}) \cdot \frac{(3, 2, 6)}{7} \frac{dx dy}{|\cos \alpha|}$$

$$= \int_0^2 \int_0^{3 - \frac{3x}{2}} \left[\frac{3xy}{7} + \frac{2}{7}y^2 + \frac{6}{7} \left(y - \frac{yx}{2} - \frac{y^2}{3} \right) \right] \frac{7}{6} dy dx$$

$$\vec{n} \cdot \vec{k} = \cos \alpha = \frac{6}{7}$$

$$\begin{aligned}
 2. (cont) \int_{IV} F \cdot dS &= \int_0^2 \int_0^{3-\frac{3}{2}x} \left(\frac{xy}{2} + \frac{y^2}{3} + y - \frac{yx}{2} - \frac{y^2}{3} \right) dy dx \\
 &= \int_0^2 \int_0^{3-\frac{3}{2}x} y dy dx = \int_0^2 \left. \frac{y^2}{2} \right|_{y=0}^{y=3-\frac{3}{2}x} dx \\
 &= \int_0^2 \frac{\left(3-\frac{3}{2}x\right)^2}{2} dx = \left. \frac{\left(3-\frac{3}{2}x\right)^3 \left(-\frac{2}{3}\right)}{3 \cdot 2} \right|_{x=0}^{x=2} \\
 &= -\frac{1}{9} \left[\left(3-\frac{3}{2} \cdot 2\right)^3 - 3^3 \right] = \frac{1}{9} \left(\cancel{0} + 3^3 \right) \\
 &= \cancel{0} + 3 = 3
 \end{aligned}$$

Bonus $\text{div } \vec{F} = y + 2y + y = 4y = \text{div } \vec{F}$

Solid Tetrahedron: $\iiint_{\Omega} \text{div } \vec{F} dV = \iiint_{\Omega} 4y dz dy dx$

$0 \leq x \leq 2$
 $0 \leq y \leq 3 - \frac{3}{2}x$
 $0 \leq z \leq 1 - \frac{x}{2} - \frac{y}{3}$

$$\begin{aligned}
 &= \int_0^2 \int_0^{3-\frac{3}{2}x} \left. 4y z \right|_{z=0}^{z=1-\frac{x}{2}-\frac{y}{3}} dy dx \\
 &= \int_0^2 \int_0^{3-\frac{3}{2}x} 4y \left(1 - \frac{x}{2} - \frac{y}{3}\right) dy dx = \int_0^2 \int_0^{3-\frac{3}{2}x} \left(4y - 2xy - \frac{4y^2}{3}\right) dy dx \\
 &= \int_0^2 \left. \left(2y^2 - xy^2 - \frac{4y^3}{9}\right) \right|_{y=0}^{y=3-\frac{3}{2}x} dx = \int_0^2 \left(2\left(3-\frac{3}{2}x\right)^2 - x\left(3-\frac{3}{2}x\right)^2 - \frac{4\left(3-\frac{3}{2}x\right)^3}{9} \right) dx \\
 &\quad \left(3-\frac{3}{2}x\right)^2 = \left(9 - 9x + \frac{9}{4}x^2\right)
 \end{aligned}$$

2. Bonus (cont)

$$\iiint \operatorname{div} \vec{F} dV = 2 \int_0^2 \left(3 - \frac{3}{2}x\right)^2 dx - \int_0^2 \left(9x - 9x^2 + \frac{9x^3}{4}\right) dx$$

$$\Downarrow \quad \quad \quad \neq \frac{4}{9} \int_0^2 \left(3 - \frac{3}{2}x\right)^3 dx$$

$$= 2 \left(-\frac{2}{3}\right) \left(\frac{3 - \frac{3}{2}x}{3}\right)^3 \Big|_{x=0}^{x=2} - \left(\frac{9x^2}{2} - \frac{9x^3}{3} + \frac{9x^4}{16}\right) \Big|_{x=0}^{x=2}$$

$$\neq \frac{4}{9} \left(-\frac{2}{3}\right) \left(\frac{3 - \frac{3}{2}x}{4}\right)^4 \Big|_{x=0}^{x=2}$$

$$= -\frac{4}{9} (0^3 - 3^3) - [18 - 24 + 9] + \frac{2}{27} (0^4 - 3^4)$$

$$= 4 \cdot 3 - 3 \neq 2 \cdot 3 = 12 - 3 - 6 = \boxed{3}$$

(This is expected by the divergence theorem which we will study this week:

$$\iiint_{\text{solid}} \operatorname{div} \vec{F} dV = \iint_{\text{boundary of the solid}} \vec{F} \cdot \vec{n} dS$$

\vec{n} \swarrow outward normal

3. Let $\vec{E} = \nabla \left(\frac{-1}{|\vec{R}|}\right)$ $|\vec{R}| = \sqrt{x^2 + y^2 + z^2}$

(a) \vec{E} is a conservative field on $\mathbb{R}^3 \setminus \{origin\}$ since is a gradient of a potential function $\frac{-1}{|\vec{R}|}$

$$\int_C \vec{E} \cdot d\vec{R} = \int_C \nabla \left(\frac{-1}{|\vec{R}|}\right) \cdot d\vec{R} = \frac{-1}{|\vec{R}|} \Big|_{(1,1,1)}^{(2,1,0)}$$

$$= -\frac{1}{\sqrt{2+1+0}} + \frac{1}{\sqrt{1+1+1}} = 0$$

3. (cont) (b) Compute $\iint_S \vec{E} \cdot d\vec{S}$ over

sphere $x^2 + y^2 + z^2 = 4$

spherical coordinates
 $r = 2 = |R|$
 $0 \leq \theta < 2\pi$
 $0 \leq \phi \leq \pi$

$$\vec{E} = -\nabla\left(\frac{1}{|R|}\right) = +\frac{R}{|R|^3}$$

$$\frac{\partial R}{\partial \phi} = (2\cos\phi\cos\theta, 2\cos\phi\sin\theta, -2\sin\phi)$$

$$\frac{\partial R}{\partial \theta} = (-2\sin\phi\sin\theta, 2\sin\phi\cos\theta, 0)$$

$$\begin{cases} x = 2\sin\phi\cos\theta \\ y = 2\sin\phi\sin\theta \\ z = 2\cos\phi \end{cases}$$

$$\begin{aligned} \frac{\partial R}{\partial \phi} \times \frac{\partial R}{\partial \theta} &= \begin{pmatrix} 2\cos\phi\cos\theta & 2\cos\phi\sin\theta & -2\sin\phi \\ -2\sin\phi\sin\theta & 2\sin\phi\cos\theta & 0 \end{pmatrix} = +4\sin^2\phi\cos\theta \vec{i} \\ &\quad + 4\cos\phi\sin\phi(\cos^2\theta + \sin^2\theta) \vec{k} \\ &= 4\sin\phi \underbrace{[\sin\phi\cos\theta \vec{i} + \sin\phi\sin\theta \vec{j} + \cos\phi \vec{k}]}_{\vec{n}} \end{aligned}$$

On the sphere

$$\vec{E} = +\frac{R}{2^3} = +\frac{2}{2^3} (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi) = -\frac{1}{4} \vec{n}$$

$$\begin{aligned} \therefore \iint_S \vec{E} \cdot d\vec{S} &= \iint_S +\frac{1}{4} \underbrace{\vec{n} \cdot \vec{n}}_{=1} (\cancel{4}\sin\phi) d\theta d\phi \\ &= + \int_0^\pi \int_0^{2\pi} \sin\phi d\theta d\phi = + \int_0^\pi 2\pi \sin\phi d\phi \\ &= 2\pi \cos\phi \Big|_{\phi=0}^{\phi=\pi} = 2\pi [+1 + 1] = \boxed{+4\pi} \end{aligned}$$

Note: Verify that $\text{div } \vec{E} = 0$ However here the div. Theorem doesn't apply because \vec{E} is not defined at the origin which lies inside the sphere!

$$4. (a) \quad \nabla^2 f = \nabla \cdot \nabla f = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial f}{\partial \rho} \right) + \frac{1}{\rho} \frac{\partial}{\partial \theta} \left(\frac{1}{\rho} \frac{\partial f}{\partial \theta} \right) + \frac{\partial^2 f}{\partial z^2}$$

$= \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \theta^2}$

since $\vec{F} = \nabla f$, $F_\rho = \frac{\partial f}{\partial \rho}$, $F_\theta = \frac{1}{\rho} \frac{\partial f}{\partial \theta}$, $F_z = \frac{\partial f}{\partial z}$

(b) Express heat equation $\frac{\partial U}{\partial t} = \kappa \nabla^2 U$ in cyl. coords.
 if U is independent of $z \Rightarrow \frac{\partial U}{\partial z} = 0$.

$$\frac{\partial U}{\partial t} = \kappa \left[\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial U}{\partial \rho} \right) + \frac{1}{\rho} \frac{\partial}{\partial \theta} \left(\frac{1}{\rho} \frac{\partial U}{\partial \theta} \right) \right]$$

$= \frac{1}{\rho^2} \frac{\partial^2 U}{\partial \theta^2}$

(c) $U(t, \rho, \theta, z) = \cos t \cos(\rho \cos \theta)$ $\kappa = 1$

$$\frac{\partial U}{\partial t} = -\sin t \cos(\rho \cos \theta)$$

$$\frac{\partial U}{\partial \rho} = \cos t (-\sin(\rho \cos \theta) \cdot \cos \theta)$$

$$= -\cos t \cos \theta \sin(\rho \cos \theta)$$

$$\frac{\partial U}{\partial \theta} = \cos t (-\sin(\rho \cos \theta) (\rho \sin \theta))$$

$$= -\rho \cos t \sin \theta \sin(\rho \cos \theta)$$

$$\frac{\partial^2 U}{\partial \rho^2} = -\cos t \cos^2 \theta \cos(\rho \cos \theta)$$

$$\frac{\partial}{\partial \theta} \left(\frac{1}{\rho} \frac{\partial U}{\partial \theta} \right) = -\cos t \sin^2 \theta \cos(\rho \cos \theta)$$

$$\frac{\partial}{\partial \rho} \left(\rho \frac{\partial U}{\partial \rho} \right) = \frac{\partial U}{\partial \rho} + \rho \frac{\partial^2 U}{\partial \rho^2} = -\cos t \cos \theta \sin(\rho \cos \theta) + \rho (-\cos t \cos \theta \cos(\rho \cos \theta))$$

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial U}{\partial \rho} \right) = -\frac{1}{\rho} \cos t \cos \theta \sin(\rho \cos \theta) - \cos t \cos^2 \theta \cos(\rho \cos \theta)$$

doesn't work
 Forget it
 Sorry