Modern Geometry

AN INTEGRATED FIRST COURSE

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McGRAW-HILL BOOK COMPANY, INC.

New York Toronto London

1958

EXERCISES

1. Give an indirect proof of the Euclidean theorem:
If the bisectors of two interior angles of a triangle are equal, the triangle is isosceles. *Hint:* See [59, p. 141]. How does the indirect proof compare in simplicity with the direct proof?

2. Are the words "invalid" and "false" equivalent in meaning? Explain.

3. Give a direct proof of the Euclidean theorem proved in Sec. 1.8. *Hint:* Through the mid-point O of MN (Fig. 1.6), draw a perpendicular to CD, meeting AB and CD in the respective points E and F. Show that the right triangles EOM and ONF are congruent.

Determine which of the following arguments are valid:

4. If Jay is pitching, our team is winning. Our team is winning; therefore Jay is pitching.

5. If Mr. X is President, he is a Democrat. Mr. X is President; therefore Mr. X is a Democrat. (If Mr. X is the present President of the United States, is the conclusion true?)

6. Good canned peaches are expensive, and this can of peaches is good; therefore this can of peaches is expensive.

7. No undergraduates have B.A. degrees. No freshmen have B.A. degrees. Therefore freshmen are undergraduates.

Concluding Remarks

The material here presented is of the selective type, since an exhaustive study of the foundations of geometry is not the primary aim of this work. However, even this brief introduction, supplemented by the discussions in the next chapter, will enable the reader to understand, appreciate, and even anticipate the great changes which have taken place in geometric thinking since Euclid gave to the world his first-class model of a logical system.

SUGGESTIONS FOR FURTHER READING*

Bell, E. T.: "The Search for Truth."
Bentley, A. F.: "Linguistic Analysis of Mathematics."
Black, Max: "The Nature of Mathematics."
Carnap, Rudolf: "Foundations of Logic and Mathematics."
Cohen, M., and E. Nagel: "Introduction to Logic and the Scientific Method."
Columbia Associates of Philosophy: "Introduction to Reflective Thinking."
Keyser, C. J.: "Human Worth of Rigorous Thinking."
Russell, Bertrand: "Introduction to Mathematical Philosophy."
———: "Mysticism and Logic and Other Essays."
Stabler, E. R.: "An Interpretation and Comparison of Three Schools of Thought."
———: "An Introduction to Mathematical Thought."
Tarski, A.: "An Introduction to Logic."
Weyl, Herman: "Mathematics and Logic."
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* For complete publication data, see the Bibliography.

Chapter 2

SPECIAL TOPICS OF AXIOMATIC GEOMETRY

Just as it would be hard to visualize a new, sleek convertible car by examining its various parts, unassembled, so too it is difficult to grasp the idea of a logical system by studying only its component parts, i.e., *undefined elements, axioms,* and *logical reasoning.* It is time now to look at some assembled products.

Euclidean geometry is perhaps one of the most famous examples of a mathematical (logical) system, but there are simpler ones which show the interrelation of the various parts without the added complications of this classical system. A study will be made first of one of these simple systems.

2.1. A Simple Logical System and a Finite Geometry

The elements of a logical system are an ornament S consisting of a set of beads arranged on wires in accordance with the following conditions:

1. Each pair of wires has at least one bead in common.
2. Each pair of wires has not more than one bead in common.
3. Every bead in S is on at least two wires.
4. Every bead in S is on not more than two wires.
5. The total number of wires in S is four.

What does the ornament look like? The answer to this question will be found by deductive reasoning.

A first conclusion, following immediately from statements 1 and 2, is:

6. Every pair of wires in S has one and only one bead in common.

From statements 3 and 4 follows the next conclusion:

7. Every bead in S is on two and only two wires.

In less formal and more familiar language, statements 6 and 7 declare that through two different beads there passes one and only one wire and that there are not more than two wires through each bead.

The number of beads in the ornament will be determined next. As a result of statement 5, there are exactly four wires in the ornament. Let (i,j) denote the bead common to the wires W_i and W_j. Then, since two things can be selected from four in six different ways, there are six beads in the ornament. Denote them by $A(1,2)$, $B(1,3)$, $C(1,4)$, $D(2,3)$, $E(2,4)$, $F(3,4)$.

By statement 2 these six beads are all distinct, and another conclusion is:

8. There are exactly six beads in S.

It is noted next that on each wire there is a bead common to (i.e., that lies on) each of the remaining wires. Thus, on W_1 are the beads (1,2), (1,3), (1,4), and hence another conclusion is:

9. There are three beads on each wire.

If now beads are represented by points and wires by curved lines, an ornament S satisfying the nine statements just given is shown in Fig. 2.1, and the power of logical reasoning has been demonstrated.

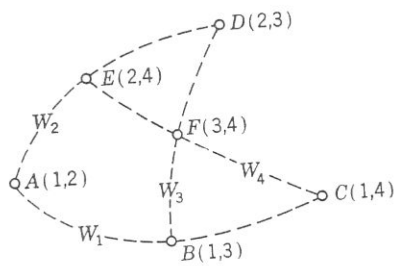

Fig. 2.1

A definition is introduced next. *Parallel beads are those not connected by a wire.* For example, points C and D of Fig. 2.1 are parallel beads. From this definition follows the next conclusion:

10. Each bead in S has exactly one bead parallel to it.

To prove this last statement, let a bead, say A, lie on the two wires W_1 and W_2. The remaining two wires then determine exactly one bead F, and no wire passes through both the beads A and F.

The logical system just described and represented in Fig. 2.1 may be linked with axiomatic geometry by replacing the words ornament, bead, wire by the geometric terms shown in column I of the table below.

	I	II
Ornament	Plane	Plane
Bead	Point	Line
Wire	Line	Point

Statements 1 to 5 then become:

a.1. Each pair of lines in the plane has at least one point in common.

a.2. Each pair of lines in the plane has not more than one point in common.

a.3. Every point in S is on at least two lines.

a.4. Every point in S is on not more than two lines.

a.5. The total number of lines in S is four.

A geometry based on these five statements, which we may now term axioms, is a highly restricted one called a finite geometry, because there are only a finite number of points on each line. The latter statement is a consequence of the fact that there were only a finite number of beads on each wire of ornament S and of the fact that logical reasoning is independent of the nature of the elements satisfying axioms of a logical system. Whether one reasons about beads on a wire or points on a line is immaterial.

Another finite geometry could be obtained by making the replacements of terms shown in column II of the table just given. Conclusion 10 would then read:

Each line in S has exactly one line parallel to it.

EXERCISES

1. How many sets of parallel beads are there in the ornament S of Fig. 2.1? Name them.

2. Does the figure consisting of the four points and four lines shown in Fig. 2.2 satisfy the five axioms just given? Give reasons for your answer.

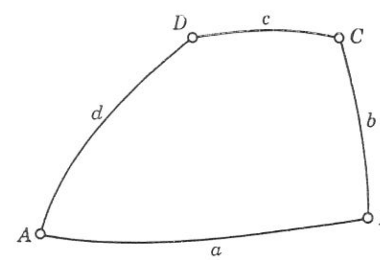

Fig. 2.2

3. Consider a system S whose undefined elements, *point* and *line*, satisfy the following axioms:

b.1. There exist exactly three distinct points.

b.2. Two distinct points determine a line.

b.3. Not all points are on the same line.

b.4. Two distinct lines determine at least one point.

Show that Axioms *b*.1 to *b*.4 are satisfied when points are taken as the symbols A, B, C and lines are vertical columns in the table below:

$$\begin{matrix} A & B & C \\ B & C & A \end{matrix}$$

Prove that two distinct points determine at most one line.

4. Suppose that a group of politicians have assembled in a room S of New York City and have formed committees for campaign work in accordance with the following instructions, called axioms:

Axioms

c.1. If A_1 and A_2 are distinct (different) men of S, there exists at least one committee containing both A_1 and A_2.

c.2. If A_1 and A_2 are distinct men of S, there is not more than one committee containing both A_1 and A_2.

c.3. Any two committees have at least one man of S in common.

c.4. There exists at least one committee.

c.5. Every committee contains at least three men of S.

c.6. Not all men serve on the same committee.

c.7. No committee contains more than three men of S.

Prove that *there are only seven men in the room. Hint:* See [75, pp. 38–42].

5. Replace the words "men" and "committee" by the respective words "point" and "line" in each of the axioms of Exercise 4. Then there results a finite geometry which is represented graphically in Fig. 2.3. Prove that in this geometry:

(*a*) Each line contains three points and on each point are three lines.

(*b*) S contains only seven points and seven lines. (Points 2, 3, 5 lie on the dotted line of the figure.)

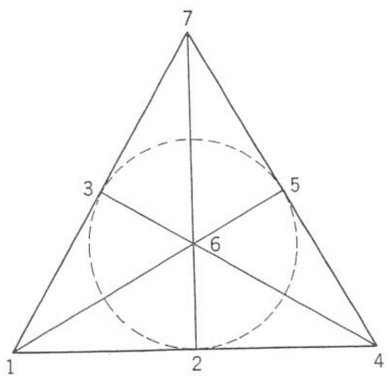

FIG. 2.3

6. Replace Axiom *c*.7 of Exercise 4 by the new axiom: "No committee contains more than four men of S." How does this replacement affect the number of men in S? *Hint:* See [67, Vol. 1, p. 6].

2.2. Concerning the Selection of Axioms

What dictates the axioms of a logical system? In the system just described, it was a physical ornament. Euclidean geometry too had its practical origin. There now seems little doubt that Euclid's purpose in compiling his "Elements" was to derive properties of physical space from a few explicitly stated definitions and assumptions. To this end, he started with 10 assumptions (see Appendix A) concerning the undefined elements, point, line, and plane; but it was their physical counterparts, i.e., a dot, a ray of light, and a smooth mirrorlike surface, which helped him to formulate these axioms.

For example, the axiom which says that a straight line may be drawn between two points is a highly idealized description of what happens when the point is replaced by a dot and the line by a taut string, ray of light, or rigid rod.

It is true that many physical lines may be drawn through two pencil dots; nevertheless, if the dots gradually become smaller and smaller, these different lines will eventually appear to coincide. By progressive abstraction, then, one finally reaches a statement concerning elements which are stripped of all physical meaning.

An immediate advantage of such a process should be obvious. Statements, or axioms, concerning undefined elements are no longer subject to those experimental errors made when dealing with physical objects, nor is the reasoning process distorted by what seems to be true of these physical elements.

2.3. Applied Geometry

It is not at all necessary that axioms of an abstract system describe properties of space or physical objects. Sheer fantasy, a powerful imagination, or a stroke of genius may be at work in the process of selecting axioms for a given system. If later, physical elements are found to have properties satisfying axioms of the abstract system, so much the better, for mathematics lives by virtue of its wide applicability. Theorems of the abstract system may then be used to describe additional and perhaps hidden properties of these physical elements. The abstract system is then said to be applied, and in such a case it is perfectly legitimate to speak of an axiom as a "self-evident fact" and of theorems as true, meaning that they are experimentally verifiable.

The application process has been illustrated in Boolean algebra, a highly abstract system which was constructed without reference to physical reality and later put to practical use in the construction of electric circuits and computing machines. The importance of these machines in modern research cannot be overestimated. The high-speed mathematical "brain," built at the Institute for Advanced Study by von Neumann, played a vital role in the hydrogen-bomb race. Calculations that would have required several lifetimes were made in a matter of months. In fact, it took this machine, believed to be the world's fastest and most accurate, six months to complete the computations on mathematical equations of the bomb.

2.4. Another Logical System: Projective Geometry

Another example of a logical system is projective geometry. It is not, as many think, an extension of Euclidean geometry but is, rather, a perfectly logical system based on sets of axioms about the undefined elements, point, line, and plane, and on two undefined relations, incidence and separation. One set of axioms, called incidence axioms, deals with the property of a point being on a line. Another set, called existence axioms, deals with the actual existence of points and lines. These and other axioms of the system will be listed later when projective geometry is studied. At this point, attention is being directed to the fact that *different sets of axioms lead to different geometries.*

2.5. Properties of a Set of Axioms

It is now generally agreed that, if a set of axioms is to lead to results of any importance in either an abstract or an applied sense, the set should be *consistent, complete,* and *independent.*

A set of axioms is said to be consistent if no two statements of the system contradict each other or if, of any two contradictory statements in the system, at least one cannot be proved.

A set of axioms is called complete if, of any two contradictory statements involving terms of the system, at least one statement can be proved in the system.

A set of axioms is said to be independent if it does not contain a single superfluous statement, i.e., a statement which can be deduced from the remaining axioms and which might therefore be counted among the theorems.

If the logical system is a geometry which is to be applied, it is important that axioms correspond well with properties of physical objects representing the undefined terms. In this way one arrives at theorems further describing these elements.

The questions of consistency of a geometry and of the independence of its axioms are, of course, related ones. To show that an axiom y of a system S is *independent* of the other axioms of the system, it is sufficient to show that these residual axioms and a new axiom y' that directly contradicts axiom y form a *consistent* system S'. For, if axiom y were not independent of the other axioms of system S, it could be deduced as a theorem. The new system S' would then contain both axiom y and a contradiction of this axiom and hence be inconsistent.

As for consistency, that is still a highly controversial question. Great and serious efforts have been made in recent time to find consistency proofs, at least for axioms of algebra, but complete success in this direction has not yet been attained.

2.6. Logical Defects in Euclid

Despite Euclid's unquestioned ability, his geometry does not satisfy the present-day requirements for logical rigor. It contains flaws. To begin with, tacit assumptions are not permitted in a logical system; yet Euclid made quite a few of them. For example, he used the assumption that a line is infinite in extent in some of his proofs, but no such assumption was made in his axioms, nor is this property of a line a consequence of his other axioms. A line can be extended indefinitely in either direction, but this does not mean that the line is infinite. A geometry will be studied later in which a line may be extended indefinitely in either direction and still be finite in length (see Chap. 15).

Again, in his proof by superposition, Euclid gives evidence of assuming facts not so stated in his axioms. The assumptions concerning congruence form an important part of any system of axioms for geometry. Lack of appreciation of this fact lies at the root of the difficulty involved in the method of superposition. In proving the congruency of two triangles having two sides and the included angle of the one equal to two sides and the included angle of the other, Euclid actually regarded one triangle as being moved in order to make it coincide with the other. He thereby tacitly assumed motion of figures without their deformation, and completely ignored the fact that points are undefined elements. If, on the other hand, one considers geometry as an applied science in which figures are capable of displacement, or motion, there cannot be ignored the modern physical notion that the dimensions of bodies in motion are not the same as when they are at rest. Relativity theory has shown that space and time cannot be separated.

When Euclid constructed lines and circles to prove the existence of certain figures, he tacitly assumed their intersection points. In a rigorous development, the existence of these points must be either proved or guaranteed by means of an axiom. Euclid's failure to do this was later corrected by an axiom (see Sec. 6, Appendix B), which ascribes to all lines and circles that characteristic called *continuity*.

Another main defect in Euclid's system was his almost complete disregard of such notions as the *two sides of a line* and the *interior of an angle*. Without the clarification of these ideas, absurd consequences result such as the paradox discussed in the next section.

2.7. A Paradox

Proof will now be given of the following theorem:

Theorem 2.1 (*A Paradox*)
Every triangle is isosceles.

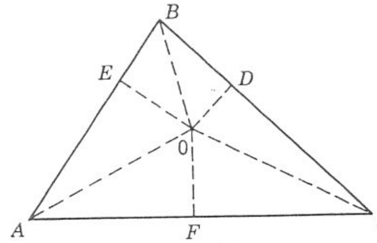

FIG. 2.4

Theorems upon which the proof is based will be found in Appendix A.

In the triangle ABC (Fig. 2.4) let O be the point of meeting of the perpendicular bisector OD of side BC and the bisector AO of angle A. Draw line OE perpendicular to side AB and line OF perpendicular to side AC. Join O to the points B and C. Then, the right triangles AOF and AOE are congruent, and hence

$$OE = OF \quad \text{and} \quad AE = AF$$

Now BO = OC

The right triangle OCF is therefore congruent to the right triangle OBE, and hence $EB = CF$. Consequently

$$AE + EB = AB = AF + FC = AC$$

and the triangle ABC is isosceles.

To explain the paradox, try constructing your own figure and see the position of point O with respect to the triangle ABC. Does it fall within or without the triangle? (See the following exercise.)

EXERCISE

Show that point O of Fig. 2.4 falls without the triangle ABC. Does $AB = AC$ in this case? Prove your answer.

2.8. Hilbert's Axioms for Euclidean Geometry

Primarily through the works of such men as Pasch, Peano, and Hilbert, logical defects in Euclid have been removed and the whole of Euclidean geometry placed on a sound, logical basis.

About the best of the modified sets of axioms for Euclidean geometry is Hilbert's (see Appendix B), first published in his book "The Foundations of Geometry," in which he considers a class of undefined elements called points and certain undefined subclasses of these points called straight lines and planes. His axioms concerning these elements are divided into the five subsets:

1. Incidence axioms
2. Order axioms
3. Congruency axioms
4. An axiom of parallels
5. Axiom of continuity

From them, all theorems of elementary Euclidean geometry may be deduced.

The assumptions, in which congruence is taken as one of the fundamental undefined notions, were probably suggested by the extensive controversies which took place as to whether Euclid regarded the idea of congruence or the idea of motion as fundamental.

Order axioms eliminate the possibility of paradoxes such as the one just given.

2.9. Concerning a Revision of an Order Axiom

In an early edition of his work "The Foundations of Geometry," Hilbert gave an order axiom containing *more assumptions* than are

found in the corresponding axiom of the seventh edition of the same work. The reasoning which dictated this change is to be studied here. It is an excellent example of a more complicated type of logical reasoning than any presented thus far. By fine detailed workmanship, a conclusion is reached which permits a simplification of the original axiom.

Two sets of axioms will be employed in the simplification process, and three axioms of the first set are:

Incidence Axioms

I.1. *Given any two points A, B, there exists a line (a) lying on A and B.*

I.2. *Given A, B, there exists at most one line (a) lying on A, B.*

I.3. *There are at least two points which lie on a given line. There are at least three points which do not lie on a line.*

The second set of axioms is concerned with order relations. Each of these order axioms is stated and carefully analyzed.

Order Axioms

O.1. *If the point B is between A and C, then A, B, C are three different points of a line, and B is also between C and A.*

This axiom implies that the term "between" is used only for points on a line and states that the relative position of points A and C does not affect B's property of lying between A and C.

O.2. *Given points A and B on a line, there exists at least one point C such that B lies between A and C.*

This axiom guarantees the existence of at least three points on a line and allows one to refrain from setting stronger existence postulates in the incidence axioms. Note, in this connection, Axiom I.3, which guarantees the existence of three points, but not all on a line.

O.3. *Given points A, B, C on a line, then at most one of these points lies between the two others.*

This axiom together with Axioms O.1 and O.2 permit the following definition:

An interval consists of those points B of line AC for which B is between A and C, and B is called an interior point of the interval.

O.4. (*Pasch's Axiom.*) *If a line α intersects one side AB of a triangle (Fig. 2.5) in a point X between A and B and if α does not pass through C, then there exists on α a point Z between C and B, or a point Y between A and C.*

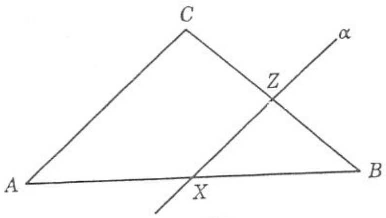

Fig. 2.5

From Axiom O.1, it is seen that α cannot pass through A or B; but, thus far, it is not known whether there exists any point X between A and B. Axiom O.2 assumes only the existence of a point on AB extended. The next theorem disposes of this question.

Theorem 2.2 (*Existence Theorem*)

If A and B are any two distinct points on a line, there exists a point X between A and B.

PROOF. From Axiom I.3, there exists a point E outside of the line AB (Fig. 2.6) and, by Axiom I.1, a line AE connecting the points A and E. Draw the line AE. According to Axiom O.2, there exists a point F on AE and a point G on FB such that E is between F and A and B is between G and F. Draw lines GE and FB.

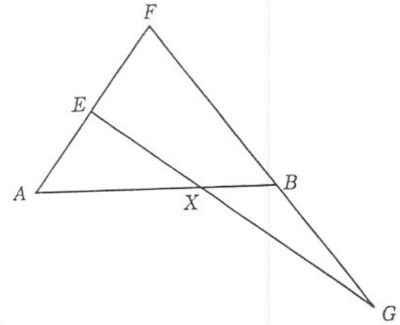

Fig. 2.6

Line GE does not pass through any of the points A, B, F. If it passed through A, point E would coincide with A, since EG intersects AF in E and therefore cannot intersect this line in a second point (Axiom I.2); if GE passed through B, it would be the line BGE and would intersect AF in F and not E. The same would be true if GE passed through F.

Now, since E is a point between A and F, Pasch's axiom may be applied to the triangle ABF. According to this axiom, GE must meet either line AB in a point X between A and B or line FB in a point Y between F and B. But the point Y cannot exist, since the intersection of lines FB and GE is, by construction, the point G; and by Axiom O.3, point G cannot lie between F and B, since B lies between F and G. Therefore line GE intersects AB in a point X between A and B, and the theorem is proved.

Corollary 1

If two points P and P′ are on the same side of a line α and if point Q is on the same side of α as P, then Q and P′ are also on the same side of α.

PROOF. If line α (Fig. 2.7) were to intersect $P'Q$, it would follow from Pasch's axiom that it would also intersect PP' or PQ in an interior point.

Corollary 2

If P and P′ are points on different sides of α and if point Q is on the same side of α as P, then P′ and Q are on different sides of α.

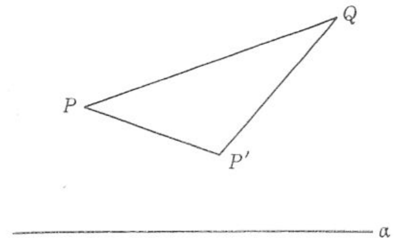

Fig. 2.7

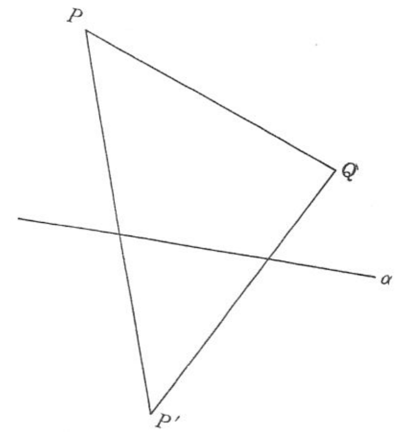

Fig. 2.8

PROOF. By Pasch's axiom, α must intersect QP' (Fig. 2.8) in an interior point, since, by hypothesis, it intersects PP' but not PQ.

Attention is called now to the distinction between Axiom O.2 and the existence theorem just proved. In the former, a point C is assumed to exist on the line segment AB extended, and it can then be proved that there exists a point on AB between A and B. Originally the second order axiom read as follows:

If A and C are two points of a straight line, then there exists at least one point B between A and C and at least one point D so situated that C lies between A and D (*Fig.* 2.9).

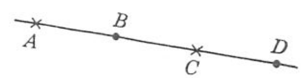

Fig. 2.9

In the light of the existence theorem just proved, this original axiom contained an assumption which was a logical consequence of the other axioms. It was therefore removed, and the assumption became a theorem. This means that Hilbert's original set of order axioms was not *independent*.

EXERCISES

1. When is one axiom of a set said to be independent of the others?
2. When is a set of axioms said to be consistent?
3. In what sense are independence and consistency properties related?
4. How can one axiom be proved to be independent of others of a set? *Hint:* See [75, pp. 69–70].
5. What is meant by a paradox?
6. Name two defects in Euclid's early system.
7. What is meant by applied geometry?
8. Is it true that, through a point P not on a line L, there is only one parallel to a line? Explain.

2.10. Euclidean Geometry and the Physical Universe

Early Egyptian geometry, it will be recalled, arose from man's experience with physical objects and was, consequently, a collection of useful facts. For example, to form a right angle, the Egyptians took a closed circle of rope and divided it by means of three knots into three parts whose lengths were in the ratio 3:4:5. Then three men, each holding a knot, stretched the rope tight, thus forming a triangular figure, as shown in Fig. 2.10.

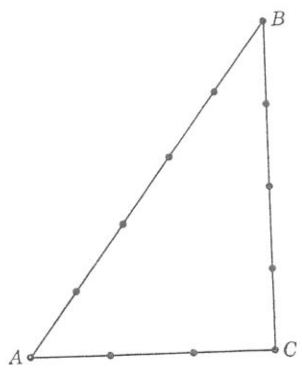

Fig. 2.10

In this way, east and west lines were drawn on the earth's surface after north and south had been determined by astronomical observations. The perpendicular lines thus obtained were probably used by the Egyptians as guides in constructing their pyramids and temples.

There were also numerical formulas for finding certain areas. For instance, the area of a flat field in the form of an isosceles triangle was found by multiplying the length of one of the equal sides by one-half the length of the base; and the area of a circular plot with a radius of r feet was taken to be

$$\tfrac{256}{81}r^2 = 3.16049r^2$$

These and the other formulas developed by practical methods sufficed for the ancient Egyptians, but not for the more advanced Greeks. With considerable wealth at their command and with slaves to do their menial work, the Greeks had ample opportunity for developing their superb talents for abstract reasoning. No more perfect target could be found than these crude formulas of the ancient Egyptians. By

forming abstract ideas of such things as points, lines, and planes, stating axioms about these elements, and reasoning logically from these axioms, the Greeks developed their abstract science.

Unfortunately, this linking of geometry, the experimental science, with geometry, the logical abstract science, led scientists and the world in general to the erroneous belief that Euclid's geometry was simply the abstract, mathematical formulation of the laws of the universe and hence that there could exist one and only one geometry.

Many examples seem to support this belief, even today. An actual flat triangular lot is supposed to have the properties of its mathematical idealization, i.e., a figure bounded by three lines. So, in the triangular lot, as in the mathematical triangle, the greatest angle lies opposite the greatest side, the sum of its angles is 180°, and if the triangular lot is a right triangle, the square of its hypotenuse is equal to the sum of the squares of the other two sides, as in a theorem of Euclidean geometry. In this, and in many other ways, the world applies Euclidean geometry to the physical universe.

Are all such applications mute evidence of the Euclidean character of space? The answer is no. If careful experiments should seem to indicate, for instance, that the hypotenuse of a right-triangular lot is 5 yards long, when the other two sides are 3 and 4 yards, respectively, that would not be convincing evidence of the Euclidean character of space; for measurements are only approximations, and small errors in measurements cannot be detected. In fact, it is now known that Euclidean geometry is only a first approximation to the geometry of the physical universe. It is highly inadequate for many of the theoretical investigations of the twentieth-century scientists.

SUGGESTIONS FOR FURTHER READING

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———: "Space, Time and Gravitation."
Forder, H. G.: "The Foundations of Euclidean Geometry."
Heath, T. L.: "The Thirteen Books of Euclid's Elements."
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