Modern Geometry

AN INTEGRATED FIRST COURSE

Claire Fisher Adler

ASSOCIATE PROFESSOR OF MATHEMATICS
NEW YORK UNIVERSITY

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Chapter 1

LOGICAL SYSTEMS AND BASIC LAWS OF REASONING

Because of the vast amount of material now classified as geometry and because of the hundreds of years in which this material has been accumulating, it is impossible in discussing modern theories to do justice to all phases of the science. Yet, to specialize is to lose the perspective necessary for gaining a proper understanding of the nature, the scope, and the importance of geometry in the twentieth century. For this reason, it has been deemed advisable to consider first the underlying structure or foundation of geometry. Specialization will then come later.

Attention is focused first on logical systems, the pattern for which was set over 2,000 years ago when Euclid performed the amazing feat of collecting and organizing into a logical sequence practically all the existing facts about geometry. Euclid's geometry is but one example of a logical system. Algebra, which is based on axioms about the number system, is another example. Projective geometry and elementary non-Euclidean geometries are still other examples that are to be studied here.

In a logical system, a set of elements is given. Some of these elements are undefined, and certain facts or statements called axioms are assumed in connection with these undefined elements. The body of conclusions obtained by reasoning logically from these axioms and definitions represents the content of the system.

When a geometry is developed in this manner, it is called "axiomatic geometry," and the method employed is the "axiomatic method."

The axiomatic approach to a science has spread, like fire, throughout the whole of mathematics. In studying such an approach, one learns, perhaps in the most natural manner, of the far deeper problems which lie at the foundations of all mathematics.

Each of the basic topics, *undefined elements*, *axioms*, and *reasoning*, will now be considered in greater detail.

1.1 Undefined Elements and Axioms

The basic elements of a logical system are those in terms of which all the others are to be defined. *Point* and *line* are usually the undefined elements of elementary geometry, but there exist geometries in which the undefined elements are circles and spheres, or number pairs in a plane, or even other elements, depending upon the particular type of geometry to be studied.

To say that *point* and *line* are undefined elements may be puzzling, particularly since definitions for these terms can be found in any standard dictionary. However, a definition simply gives the meaning of one word in terms of others whose meaning is already clear. Simpler words may be defined in still more simple terms, and so on. Such a process would, therefore, lead to an endless regression if it were not agreed that certain basic words are to be left undefined.

For example, a line segment is, by definition, that portion of a line lying between two given points on the line. Here, *point* and *line* are undefined, as is also the word *between*.

Euclid defined a line as "length without breadth," and a straight line as a "line which lies evenly between two of its points." Here, the terms *length*, *breadth*, and *lies evenly* are all undefined; hence Euclid could just as well have used *line* as an undefined term.

Just as certain simple elements are chosen as fundamental ones, in terms of which all others are to be defined, so too some simple statements concerning the undefined elements are chosen as fundamental, in the sense that all other statements of the system are to be deduced from them by logical reasoning. *These fundamental statements which are accepted without proof are called axioms.* Their role and significance will be brought out after the reasoning process has been analyzed.

EXERCISES

1. Why must there be undefined terms in a logical system?
2. Define a circle, explaining which of the terms used are undefined.
3. Using the undefined elements *point*, *line*, *distance*, and *angle*, define (*a*) a parallelogram, (*b*) a rhombus, (*c*) a polygon.

1.2 Inductive Reasoning

Reasoning plays an important role in everyday life. In fact, were it not for man's ability to reason, it is doubtful if he could ever have advanced much beyond the primitive stage. Certainly the keener, the more penetrating, the more inclusive the operation of the mind, the greater is the likelihood of man's being able to mold his world into an environment satisfying his needs.

Reasoning also plays a dominant role in the development of a logical system, *inductive* reasoning in discovering theorems, and *deductive* reasoning in proving them. Inductive reasoning will be discussed first.

Necessity and curiosity have at all times caused people to investigate phenomena and to attempt to find the laws governing the physical universe. The inundations of the Nile and the need for reestablishing landmarks led the ancient Egyptians to develop simple properties of right triangles. Curiosity concerning the heavens brought forth various complicated theories about the actions of the planets and the sun. Cures are still being sought for the dread disease cancer.

Very much the same type of reasoning is used in all of these investigations. A doctor, for instance, arrives at a diagnosis of a specific disease by a careful, systematic investigation of all factors attendant upon the disease. He notes all symptoms, however trivial, excluding none until it has been proved irrelevant. He classifies, examines, and combines pertinent facts until he finally reaches the diagnosis which enables him to effect a cure. Reasoning of this kind is *inductive*.

Often, to verify a conclusion reached by inductive reasoning, the investigator makes repeated experiments. Sometimes they are carried on for a period of years, as was the case in establishing the laws of astronomy and the laws of heredity. Galileo also used inductive reasoning when he repeatedly dropped objects from the top of the leaning tower of Pisa to determine his famous law of falling bodies: $s = \frac{1}{2}gt^2$, where s is the distance in feet a body falls in t seconds and g is the gravitational constant whose value is about 32.2 feet per second.

Many other examples may be cited. There is, for instance, the constantly recurring phenomenon of the sun's rising and setting each day, from which it has been concluded that the sun will rise and set every day in the future. This does not mean, however, that it is absolutely certain that the sun will rise tomorrow, or a week, or a year, or a million years from now. Conclusions reached by induction are only statements of what is more or less likely to happen. If an exception is ever found, either the conclusion is discarded, or laws of probability are used to determine its reliability for future prediction.

EXERCISES

1. Give an original example of the use of induction in arriving at commonly accepted conclusions.

2. If, for the past twenty years, the average temperature in July is higher than that for December, can one conclude that the average temperature for July is always higher than that for December? Give reasons for your answer.

3. If, in each of 500 cases treated, treatment A cured disease B, can one conclude that treatment A will always cure disease B? If treatment A failed to cure disease B on the 501st treatment, should it be abandoned? Give reasons for your answer.

1.3 Some Elementary Logic

To attempt to introduce any but the most elementary principles of logic would be a formidable task, since this science, like geometry, has had a long, interesting period of growth. More than 2,000 years have elapsed since Aristotle first formulated his laws for human reasoning, and in that time radical changes have taken place. Today there is a Whitehead-Russell approach to logic, a formalist approach, headed by Hilbert, and an intuitionist approach headed by Poincaré, Weyl, and others. All have had profound effect on the foundations of mathematics, and at times the effect has been extremely disturbing. The intuitionists' view, for example, if taken literally, would have eliminated a large and important body of mathematics. Current investigations, however, are showing that the different theories are not so far apart as they seemed to be a few years ago. See [6, supplements A, B; 17, pp. 214–217; 62, pp. 247–255].*

There is space here to present only a somewhat modernized version of the classical theory.

Simple Statements and Basic Laws of Reasoning

By a statement is here meant a meaningful sentence such as

It is raining. (1)

which has for its negation, or denial, the equally simple statement:

It is not raining. (2)

Such an expression as "X is an integer" is not a meaningful sentence until X is replaced by a number.

Assuming the customary meaning of truth and falsity in a factual sense, few will deny that, if (1) is true, (2) is false. There seems to be no other possibility, and yet L. E. J. Brouwer, a famous Dutch mathematician of the twentieth century and a leader of the intuitionist school of thought, has raised some questions about this matter when applied to infinite sets.

Absurd as it may seem, Brouwer's position has not been taken with-

* Numbers in brackets refer to entries in the Bibliography.

out sufficient justification. Not all simple declarative statements can be either affirmed or denied. This, of course, is true of statements, such as, "World War I was fought to preserve democracy," whose truth or falsity might possibly be a matter of opinion or belief. It is true also of other types of statements. Suppose, for example, a man says:

I am lying.

Is his statement true? If so, he is lying, and his statement is false. Is his statement false? If so, he is lying, and his statement is true.

Again, suppose that Mr. Smith is a small-town barber who shaves those men and only those men of the town who do not shave themselves. Which of the following statements is true?

Mr. Smith shaves himself.
Mr. Smith does not shave himself.

Note the dilemma you are in if you attempt to answer this question. If you affirm the first statement and deny the second, you are admitting that Mr. Smith is shaving somebody who shaves himself, contrary to the hypothesis; if you reverse your opinion, affirming the second and denying the first, Mr. Smith is not shaving somebody who does not shave himself, and in this case, too, the hypothesis is contradicted.

Again, can you say which of the following mathematical statements has been proved to be true?

$2^{\sqrt{2}}$ *is a rational number.*
$2^{\sqrt{2}}$ *is not a rational number.*

It will be recalled that a number N is rational if there exist two integers m, n, such that

$$N = \frac{m}{n}$$

As late as the year 1941* no one had found two integers m, n satisfying the condition

$$2^{\sqrt{2}} = \frac{m}{n}$$

but neither had anyone proved that such integers did not exist. It was therefore impossible to affirm, deny, prove, or disprove either statement.

Fortunately, classical logic avoids controversial questions of such a nature by assuming in advance of an argument that any given statement is either true or false.

The three cornerstone laws of classical Aristotelian logic, called,

* See Harry Pollard, "The Theory of Algebraic Numbers," *Carus Monographs*, no. 9, p. 45.

respectively, the law of *identity*, the law of the *excluded middle*, and the law of *noncontradiction*, are:

1. *A thing is itself.*
2. *A statement is either true or false.*
3. *No statement is both true and false.*

Long-accepted patterns of reasoning are woven around these three laws, but these laws are not usable in vast regions of modern mathematics. In 1912, Brouwer challenged the second law, and a few years later Count Alfred Korzybski, a Polish-American logician, challenged the first. The third law, which deals with the consistency of a system, seems to have held its ground better than the other two laws. Still, in over 2,000 years of trying to reach an agreement on the use of these laws little has been accomplished, except the realization of the need for such an agreement.

Composite Statements

By a composite statement is meant one involving such connecting words as:

and, *or*, *if-then*

Three composite statements are:

It is raining and *John is studying.* (3)
It is raining or *John is studying.* (4)
If *it is raining,* then *John is studying.* (5)

In modern terminology, (3) is called a *conjunction*, (4) *a disjunction*, and (5) *an implication.*

The *conjunction* of any two statements p, q is the statement "*p and q,*" *which is assumed to be true when both p and q are true; otherwise, it is false.*

Thus, if p is the statement "it is snowing," and q is the statement "the wind is blowing," the conjunction "it is snowing, and the wind is blowing" is true, if it is actually snowing and at the same time the wind is blowing.

The *disjunction* of the statements p, q is the statement "*p or q,*" which *is assumed to be true when at least one of the statements is true; otherwise, it is false.*

The word "or" in this definition is used in a nonexclusive sense, in that the disjunction is still true when *both p and q* are true.

Usually, "or" is used in the exclusive sense illustrated in the statement "either I shall go, or I shall stay," in which the occurrence of one thing excludes the occurrence of the other.

From the new meaning of "or," it follows that the disjunction (4) is *true* if it is raining and John is studying; it is also true if it is not raining and John is studying; and finally, it is true if it is raining and at the same time John is not studying. It is *false*, if it is *not* raining and, at the same time, John is *not* studying.

An implication, such as (5), is a statement of the form "if p, then q" and is written symbolically $p \to q$. *Its precise meaning must be understood.*

What does it mean to say, "If it is raining, then John is studying"? If it is actually raining, the statement states unequivocally that John is studying. But, it may not be raining. If so, no information is given as to whether or not John is studying. It is possible, therefore, that John might be studying even if it were not raining; hence (5) is equivalent to the disjunction:

It is not raining, or John is studying.

and an implication $p \to q$ means that p is false *or* q is true, if "or" is now used in the nonexclusive sense.

The various possibilities of truth and falsity of the statements p, q and the implication $p \to q$ are shown in the table below, where T and F are the respective abbreviations for true and false.

p	q	$p \to q$
T	T	T
F	T	T
F	F	T
T	F	F

From this table, it is seen that the implication $p \to q$ is true for every combination of truth values of p and q except the one in which p is true and q is false. This means that a *true statement cannot imply a false one.* Thus, the implication:

If the sun shines today, $1 + 1 = 3.$

is false if the sun shines today and is true if the sun does not shine today. This is the case because $1 + 1 = 3$ is false; hence the implication is true only if the statement "the sun shines today" is false.

EXERCISES

1. Give an original example of (*a*) an implication, (*b*) a disjunction, explaining when each is true and false.

2. Check the truth or falsity of each of the following implications:
(*a*) If New York is a small city, then $2 \times 2 = 4$.
(*b*) If New York is a large city, then $2 \times 2 = 5$.
(*c*) If you are a freshman, then the grass is red.

1.4 Deductive Reasoning

Return now to the implication of the preceding section:

If it is raining, then John is studying. (1)

and, to it, add the further information:

It is raining. (2)

Then from (1) and (2) one obtains, by the fundamental rule of inference, the definite conclusion:

John is studying. (3)

and the complete process or argument by which this new statement is obtained is called *deductive reasoning*. Logical and deductive reasoning are here assumed to be synonymous terms.

There are other ways of describing the deductive process. The conclusion (3) is also said to be the inescapable consequence of the hypotheses, or, preferably, (3) is said to be a *valid* conclusion reached by a *valid* argument. It is of the syllogistic type, with its major premise (1), its minor premise (2), and its inescapable conclusion (3) formed from the nutcracker of the other two.

In general, if p and q are any two statements, this type of syllogism, represented symbolically as follows:

$$p \to q; \qquad p; \qquad \therefore q$$

says in words: If one accepts the truth of the implication $p \to q$ and also the truth of p, then one must accept the truth of q.

If "not p" denotes the denial of statement p, another equally valid argument is:

$$p \to q; \qquad \textit{not } q; \qquad \therefore \textit{not } p$$

which, in other words, says that, if one accepts the truth of the implication $p \to q$ and denies the truth of q, then one must deny the truth of p. This is so, because a true statement cannot imply a false one.

A simple example of this latter type of reasoning is:

If it is snowing, the temperature is below zero.
The temperature is not below zero.
Therefore, it is not snowing.

A mathematical argument of the same type is:

If in triangle ABC side a equals side b, then the opposite angles A and B are equal.
Angle A is not equal to angle B.
Therefore, side a is not equal to side b.

1.5 Abstract Nature of Deductive Reasoning

Form is stressed in the deductive process, rather than content of the individual statements. It makes no difference in the validity of the conclusion whether one is talking about rockets to the moon or about mere x's and y's devoid of physical meaning. It makes no difference whether the conclusion reached is true or false in a factual sense.

Many valid arguments may be given in which the conclusion is true in some instances and false in others, as the next argument shows:

If you are a member of this class, you are over twenty-one years old.
You are a member of this class.
Therefore, you are over twenty-one years old.

Even though you are actually seventeen years old, you are here bound to accept the conclusion as true, if you are a member of this class and have agreed to the original statement. There is no turning back. But this should not be a matter for concern. You have not agreed to accept actual, or factual, truth or falsity but simply the validity (or logical truth) of an abstract argument of the form:

If you are an X, you are a Y.
You are an X.
Therefore, you are a Y.

What is an X? A Y? These questions are immaterial. It is as Bertrand Russell once said, somewhat facetiously: "Mathematics is the subject in which we never know what we are talking about nor whether what we say is true." His remark makes sense when one notes that terms used in the deductive process are undefined, and conclusions rest ultimately on unproved (and sometimes meaningless) statements.

Conclusions obtained by deductive reasoning are independent of the nature of the elements involved and are completely detached from opinions, beliefs, facts, feelings, or emotions in any way connected with these elements.

1.6. Valid and Invalid Arguments. A Circle Test for Validity

By an invalid argument is meant one in which the conclusion is not a logical consequence of the hypotheses. For example, an argument in which one assumes the truth of statement p, from the truth of the implication $p \to q$ and the truth of q, is invalid. Symbolically, such an invalid argument is written:

$$p \to q; \qquad q; \qquad \therefore p$$

The following is an example of an invalid argument:

If a quadrilateral is a rhombus, its diagonals are perpendicular to each other.
The diagonals of the quadrilateral ABCD are perpendicular to each other.
Therefore, the quadrilateral ABCD is a rhombus.

That the argument is invalid is shown by reference to Fig. 1.1, where the *diamond*-shaped quadrilateral $ABCD$, which is not a rhombus, has

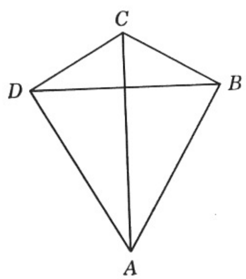

Fig. 1.1

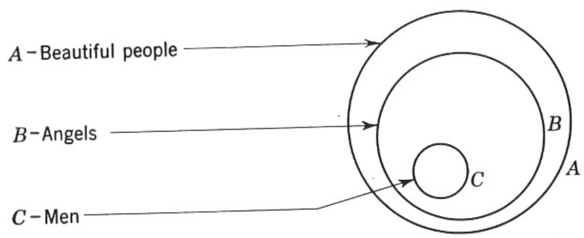

Fig. 1.2

its diagonals perpendicular. In other words, perpendicularity of its diagonals is no guarantee that a quadrilateral is a rhombus.

There is a simple circle test for determining the validity of a conclusion deduced from statements involving such words as "all," "some," "any," "every," and it will be illustrated for the following argument:

All men are angels.
All angels are beautiful.
All men are beautiful.

In applying the test, it will be assumed that two notions, (1) a set of elements and (2) belonging to a set, are known.

Points of a circle or simply a circle will represent the totality of elements of a certain set, and placing one circle inside another will show

graphically that the set of elements represented by the smaller circle belongs to, or is a subset of, the set represented by the larger circle.

Let circle A (Fig. 1.2) represent beautiful people; circle B, angels; and circle C, men. Then, from the first statement in the argument, circle C lies within circle B; and from the second, circle B lies within circle A. Circle C therefore lies within circle A, which means that men form a subset of beautiful people, and the conclusion is therefore valid.

The circle test is applied next to an argument whose validity is to be determined by means of the test.

If all good cars are expensive, and
All foreign cars are expensive, then
All foreign cars are good.

Let circle A (Fig. 1.3) *represent expensive things;* circle B, *good cars;* and *circle* C, *foreign cars.* Then, from the first statement, circle B lies

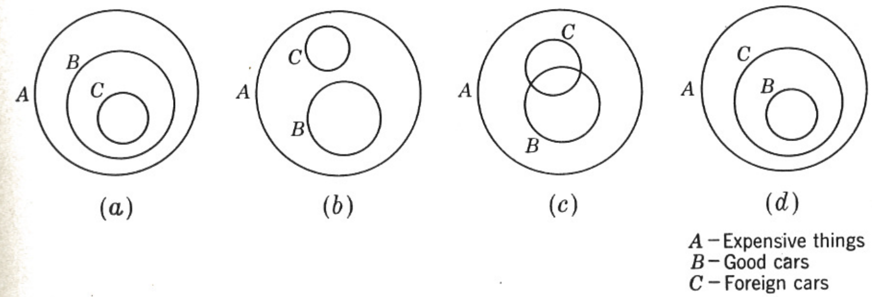

Fig. 1.3

within circle A; and from the second, circle C lies within circle A; but no information is given as to the position of circle C with respect to circle B. Circle C might lie within circle B (Fig. 1.3*a*), be external to B (Fig. 1.3*b*), overlap B (Fig. 1.3*c*), or include B (Fig. 1.3*d*). Since one is not forced to accept any particular one of these possibilities and reject the others, the conclusion is invalid.

1.7 Deductive Reasoning in an Elementary Proof

Most mathematical proofs consist of chains of simple deductions from definitions, axioms, and previous theorems. Since the syllogistic form is cumbersome, the argument is usually abbreviated, as shown in the proof of the elementary Euclidean theorem:

If two straight lines intersect, the vertical angles are equal.

The proof will be based on only definitions and axioms. Let $\angle a$ and $\angle c$ be any vertical angles (Fig. 1.4). Then, by definition of a straight angle,

$$\angle a + \angle b = \text{a straight angle} \tag{1}$$

and

$$\angle b + \angle c = \text{a straight angle} \tag{2}$$

Therefore, by the axiom that "things equal to the same thing are equal to each other,"

$$\angle a + \angle b = \angle b + \angle c \tag{3}$$

Hence, by the axiom: "If equals are subtracted from equals, the results are equal,"

$$\angle a = \angle c \tag{4}$$

Since $\angle a$ and $\angle c$ were any vertical angles (by definition of vertical angles), the theorem is proved.

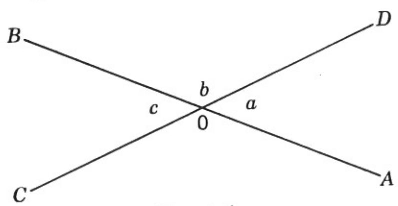

Fig. 1.4

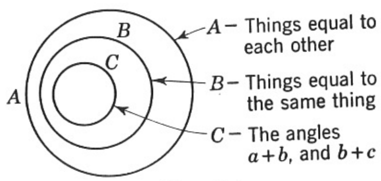

Fig. 1.5

There are two distinct syllogistic arguments in this proof. One of them leads to conclusion (3), the other to conclusion (4). The circle test will be used to check the validity of (3), and the test for (4) left as an exercise. Let circle A (Fig. 1.5) represent the set of things equal to each other; circle B, the set of things equal to the same thing; and, finally, circle C, the pair of angles $a + b$ and $b + c$. Since each of these angles is a straight angle, circle C lies within circle B, and since things equal to the same thing are equal to each other, circle B lies within circle A. Circle C therefore lies within circle A, and the validity of conclusion (3) is established.

1.8. Indirect Method of Proof

Not all proofs proceed in the direct manner just shown. Indirect proofs are extremely powerful and elegant, and yet beginners are reluctant to use them. This type of proof employs the logical principle of the excluded middle, in which an investigator assumes that either a statement or its denial is true. If one is disproved, the other follows.

For example, to show that the sum c of two even integers is an even integer, it is shown that the contrary assumption "c is odd" leads to a contradiction. By the law of the excluded middle, c is either even or odd and since c cannot be odd, then c is even.

Details of the proof, if a and b are the two even integers, are as follows:

If

$$a + b = c$$

then

$$\frac{a+b}{2} = \frac{a}{2} + \frac{b}{2} = \frac{c}{2}$$

But if each of the quantities $a/2$ and $b/2$ is an integer (which must be the case, since a and b are even), their sum is also an integer. The left-hand side of this last equation is therefore an integer; but since, by assumption, c is odd, the right-hand side $c/2$ is not an integer, and a contradiction has been reached.

The indirect method is illustrated again in the following proof of the elementary Euclidean theorem:

If two parallel lines are cut by a transversal, the alternate interior angles are equal.

Let transversal T (Fig. 1.6) cut the two given lines AB and CD in the points M and N. To show that the alternate interior angles AMN and MND are equal, it will be shown that the contrary assumption, i.e., the inequality of these angles, leads to a contradiction. It is assumed in the proof that Theorem 27, Appendix A, has already been proved.

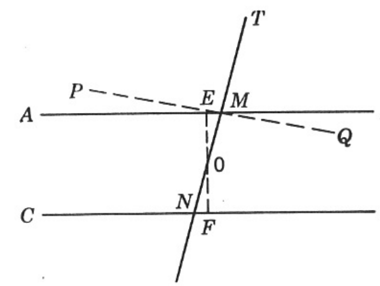

Fig. 1.6

Suppose that the angles AMN and DNM are not equal, and let a line PQ through the point M make angle PMN equal to angle DNM. Then,

PQ is parallel to CD. (Theorem 27, Appendix A)

and hence through the point M there are two parallels to the line CD. Since this conclusion contradicts the parallel axiom (Sec. 5, Appendix B), the assumption that angles AMN and DNM are unequal is false, and hence, by the law of the excluded middle, these angles are equal.

The student should compare this proof with the much longer direct proof (see Exercise 3 below).

Other examples and discussions of the indirect method may be found in the literature [12, pp. 137–153; 62, pp. 70–72]. In passing, however, it is noted that, although indirect proofs may usually be replaced by direct ones, there are some theorems which by their very nature preclude the possibility of a direct proof [17, pp. 86–87].

EXERCISES

1. Give an indirect proof of the Euclidean theorem:

If the bisectors of two interior angles of a triangle are equal, the triangle is isosceles. *Hint:* See [59, p. 141]. How does the indirect proof compare in simplicity with the direct proof?

2. Are the words "invalid" and "false" equivalent in meaning? Explain.

3. Give a direct proof of the Euclidean theorem proved in Sec. 1.8. *Hint:* Through the mid-point O of MN (Fig. 1.6), draw a perpendicular to CD, meeting AB and CD in the respective points E and F. Show that the right triangles EOM and ONF are congruent.

Determine which of the following arguments are valid:

4. If Jay is pitching, our team is winning. Our team is winning; therefore Jay is pitching.

5. If Mr. X is President, he is a Democrat. Mr. X is President; therefore Mr. X is a Democrat. (If Mr. X is the present President of the United States, is the conclusion true?)

6. Good canned peaches are expensive, and this can of peaches is good; therefore this can of peaches is expensive.

7. No undergraduates have B.A. degrees. No freshmen have B.A. degrees. Therefore freshmen are undergraduates.

Concluding Remarks

The material here presented is of the selective type, since an exhaustive study of the foundations of geometry is not the primary aim of this work. However, even this brief introduction, supplemented by the discussions in the next chapter, will enable the reader to understand, appreciate, and even anticipate the great changes which have taken place in geometric thinking since Euclid gave to the world his first-class model of a logical system.

SUGGESTIONS FOR FURTHER READING*

Bell, E. T.: "The Search for Truth."
Bentley, A. F.: "Linguistic Analysis of Mathematics."
Black, Max: "The Nature of Mathematics."
Carnap, Rudolf: "Foundations of Logic and Mathematics."
Cohen, M., and E. Nagel: "Introduction to Logic and the Scientific Method."
Columbia Associates of Philosophy: "Introduction to Reflective Thinking."
Keyser, C. J.: "Human Worth of Rigorous Thinking."
Russell, Bertrand: "Introduction to Mathematical Philosophy."
———: "Mysticism and Logic and Other Essays."
Stabler, E. R.: "An Interpretation and Comparison of Three Schools of Thought."
———: "An Introduction to Mathematical Thought."
Tarski, A.: "An Introduction to Logic."
Weyl, Herman: "Mathematics and Logic."
Whitehead, A. N.: "An Introduction to Mathematics."
Wilder, R. L.: "Introduction to the Foundations of Mathematics."

* For complete publication data, see the Bibliography.