NON-EUCLIDEAN GEOMETRY

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an overall picture of the whole subject, and in particular to prove the theorem on the angle of parallelism.

The proofs of this theorem are of two kinds:

(i) Stereometric and planimetric proofs based on the properties of horocycles, and

(ii) the so-called elementary proofs which make use only of the simplest properties of the plane.

These proofs, of which that of Straszewicz appears to be the perfect example, do not refer to any further theorems but use ingenious limiting processes and require the solution of functional equations. These I do not consider to be "accessible".

We have chosen the proof based on the study of the horosphere not only to mark the occasion of the centenary of Lobatchevsky's death but also in view of its advantages.

The present book contains three chapters. The knowledge of elementary geometry gained in schools and colleges will be sufficient for an understanding of the first two. The third chapter demands a certain familiarity with the principles of trigonometry; for §§ 28 and 29 elements of analytical geometry are also needed.

Chapter I gives some information about the history of geometry; chapters II and III contain a systematic exposition, without referring back to chapter I, of the principles of non-Euclidean geometry.

The exposition has been presented in such a way that the reader may limit himself to an examination of §§ 8-20 and obtain thereby a certain completeness of information. The remaining sections deal with non-Euclidean trigonometry.

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Warsaw, 1956

CHAPTER I

FROM THE HISTORY OF GEOMETRY

§ 1. Earliest times

Geometry probably originated in Ancient Egypt. The Greek historian Herodotus describes in the following manner how the first systematic geometrical observations were made. The inundations of the Nile, bringing with them its fertile silt, would obliterate the boundaries between properties; each year these boundaries had to be delineated anew. This task, which would be troublesome even to a modern surveyor, had to be carried out rapidly and justly. It used to be performed by specialists, whom later the Greeks referred to as "harpenodapts", i. e. ropetyers—since, apparently, their main tool was the geodetic rope (today we use the geodetic tape). More detailed information about the proceedings of the harpenodapts has not been preserved. There is no doubt, however, that constant work on the same subject must have led to a considerable familiarity with geometrical figures and to the revelation of various laws. The harpenodapts were held in high esteem by their contemporaries. Democritus, the fifth-century Greek philosopher, boasted that nobody, not even the Egyptian harpenodapts, could excel him in the art of drawing lines, testifying thereby that in his time the Egyptians still ranked high as the most skilful geometers.

In the other countries of the East, in Babylonia and Assyria, geometry was also cultivated, though perhaps to a lesser extent. During the past twenty-five years, numerous mathematical texts in the cuneiform characters have been deciphered. It appears from them that the Babylonians had developed to a considerable extent the theory of equations; they were, for instance, able to solve quadratic equations. They also knew and were applying Pythagoras' theorem, the discovery of which should consequently be placed several centuries before the birth of Pythagoras. It is impossible to decide whether Pythagoras rediscovered it or whether he merely took it from Babylonian tradition and transplanted it in Greece. What most interests us here is the fact that geometry had already started in the Mediterranean countries and penetrated from them to Greece long before the Greeks became active in that field. The credit for introducing this science was attributed by Greek historians to Thales of Miletus (sixth century B.C.), but, when we bear in mind the lively trade-relations between Greece and Egypt, he certainly cannot have been its only propagator. In the sixth century B.C. began the development of Greek geometry, shortly to flourish magnificently.

What was the standard of the Greek geometry in the sixth century? We lack records from this period. We have to depend on the accounts of authors who were writing much later and on indirect deduction. The former, for example, attribute to Thales the discovery of the theorem relating to the isosceles triangle and the vertical angle theorem, which suggests that Greek knowledge of that period was confined to simple basic principles. On the other hand, certain works have survived which bear witness to the skilful application of constructional methods. There still exists today a tunnel dug in the sixth century B.C. through a hill on the island of Samos by an architect called Eupalinus. During the construction of this tunnel, which is two-thirds of a mile long, the adits were started on both sides of the hill and met in the middle with an error that scarcely amounted to a few yards. This is an impressive result when we remember that theodolites and other instruments now used were unknown in those days. We do not know Eupalinus' procedure; he must at any rate have been acquainted with numerous geometrical properties and have been able to measure angles accurately, and to calculate accurately the difference in level between the ends of his tunnel. At all events, he proved a master of the practical application of geometry. We gather from all this that the Egyptians and their successors, the Greeks of the sixth century, had collected a considerable knowledge of geometry, especially of those aspects of it which were of practical importance in building and similar occupations.

Into all this crude and empirically collected material the incomparable Greek genius introduced logical order, transforming a conglomeration of scattered facts into a compact science which was capable of deducing one theorem logically from another. This process, of course, lasted over many generations.

It seems that the first steps in this direction were taken by Pythagoras and his pupils, known as the Pythagoreans. A Greek historian (Eudemus, as quoted by Proclus) tells us: "Pythagoras has transformed geometry by formulating all-embracing principles and developing theorems by means of pure abstract argument". Tradition considers Pythagoras to have been the first to seek clarity in the concepts used and refers to him as the originator of the idea of definition. In the Pythagorean school (fifth and sixth centuries B.C.) abstract views were conceived for the first time; namely that a geometrical line has length but no breadth, that a circle is a line all of whose points are equidistant from a fixed point, and that a tangent to a circle is that straight line which has one point only in common with it. This standpoint, that a tangent to a circle has only one point in common with it, is already a far cry from any conclusions that could be reached by direct experimental observation of real straight lines and real circles. In the fifth century B.C. it was subject to vehement criticism from Protagoras, who pointed out that a real tangent to a real circle has in common with it by no means one point, but in fact a definite segment (Fig. 1). Protagoras accused these geometrical notions of fictitiousness; he indicated that the geometry deals with objects which do not and cannot exist, i. e. with arbitrary and preposterous inventions. A science, he said, ought to examine reality—that which exists in fact. Protagoras' objections were by no means shallow ones and it is worth while to examine this question in detail.

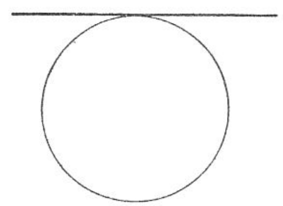

Fig. 1

A straight line drawn on a piece of paper is in fact a strip. A strip, admittedly, of minuscule width, but a strip nevertheless. The same applies to a drawn circle. Now these two lines are tangent when one strip overlaps the other as in Fig. 2.

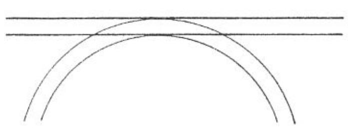

Fig. 2

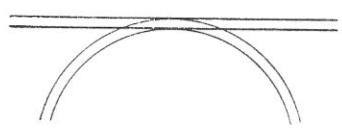

Fig. 3

These strips have, then, a common part which is obviously not a point but which has a certain "length". One might think at first glance that this fact is due simply to the imperfection of the draughtmanship and ought to vanish, or at least to be considerably diminished, with the use of more subtle drawing instruments. However, the matter is not so simple. Let us look at Figs. 2 and 3. The strips in Fig. 3 are much thinner than those in Fig. 2.

Nevertheless the common length of the strips has remained almost the same. Now if we imagine that these strips are drawn thinner and thinner we must assume at the same time that our sense of sight becomes more acute if it is to perceive these minute objects—so that the common length of the two strips will appear greater. Similarly if we examine a strand of spider's-web through a magnifying-glass we will see it quite distinctly although it may be invisible to the naked eye, but at the same time its length will also apparently increase. In other words, if we were to observe a circle and its tangent made of the finest strands of cobweb, and if our eye were able to distinguish these strands, their common length would not appear to be so small. The size of this common part should not be estimated by comparison with a fixed unit of length, a centimetre for instance, but by comparison with the width of the "strips"—that is, we should consider the ratio of the common length of the lines to their width. A piece of elementary calculus work gives here an unexpected result. It appears, in fact, that the ratio of the common length of a circle and its tangent to the "width" of the lines by no means diminishes as we draw them thinner and thinner, but distinctly increases. We cannot therefore refute Protagoras' objections by blaming the drawing instruments; we cannot assert that the tangent theorem will work with greater exactitude as we use better materials, nor can we maintain that the properties of the figures of our "practical" geometry will tend more closely to those of our "abstract" geometry as we use more and more perfect methods of draughtsmanship.

As with the circle and its tangent, we meet with the same difficulties in other geometrical problems. We state, for example, that two straight lines intersect in one point, or in other words that two arbitrary intersecting straight lines define a point. Every draughtsman will without doubt contend that two perpendicular straight lines factually determine a point; but this breaks down for straight lines that form an angle of less than ten degrees, such straight lines do not "determine" a point (Fig. 4).

FIG. 4

No—such straight lines cannot be employed for the precise definition of a point. It can be shown that if two perpendicular straight lines are taken as cutting at a "point", the straight lines in Fig. 4 cut at seven "points", and matters are not altered when the lines are drawn thinner.

In conclusion: "theoretical" geometry cannot be considered as the limiting case of the "real" one as the sizes of points and the widths of lines decrease. It is a representation of reality which is simplified in another way. Protagoras was right: there is a difference between real facts and the postulates of theoretical geometry.

§ 2. Plato

As may be inferred from the title of one of Democritus' works there developed during the fifth century B.C. a discussion around the criticism of Protagoras. We know, however, nothing about its course. But we may guess from several passages in the works of Plato, and especially of Aristotle, who was continually returning to the subject of the circle and the tangent, that the matter had aroused real interest and endless argument. Later ages, under the spell of the triumphant development of theoretical geometry and the extraordinary usefulness of its applications, somewhat slided over the fundamental speculations of Protagoras, but in the fifth and fourth centuries they were certainly not treated lightly. We possess no records by means of which we might trace

the evolution of Greek opinion, but it seems likely that the objections of Protagoras and the desire to refute them exercised an essential influence on the views of Plato (fourth century B.C.).

Generally speaking, it is difficult and in many ways controversial to characterize Plato's doctrine. Plato makes his points in a poetic and picturesque manner, using numerous suggestive comparisons; he wishes at times to draw the reader into his frame of mind, into his ardour for research, of which his dialogues are so full, rather than to communicate to him accurately and methodically the results of his enquiries. Moreover, there is nothing stiff and academic in Plato's doctrine; ideas conceived in one dialogue are modified in others—not only modified, but sometimes also caricatured, mocked and cast aside to make room for new ones. That flexibility of his views which reflects the constant evolution of thought steadily searching for the truth, but never satisfied with the results, is the reason why commentators hold to this day contradictory opinions as to Plato's theses is most principal matters. It is simply impossible to formulate these theses precisely without violating them, distorting and changing their colours.

Fortunately, for our purposes, there is no necessity to discuss the whole of Plato's philosophy, and it suffices to present his views on the relationship between theoretical and empirical geometry.

Of course, this question is only a small part of a more extensive problem, but concerning this small part Plato's standpoint, as set forth in the dialogue *Republic* and in the *Letter VII*, is quite clear and leaves no room for serious doubt. Plato admits that "the circle drawn or manufactured by man is far from our notion of a circle", since "such a circle coincides in every portion with a straight line"; he does not deduce from this, however, that geometrical theories deal with objects that have no real existence. For the subject-matter of these theories does not consist of drawn or manufactured circles and straight lines, but of ideal circles, ideal straight lines and ideal triangles, or in his own words of the "ideas" of a straight line, of a triangle or of a circle. These are by no means fictions, vain toys of the human mind, but have an objective existence independent of the human imagination, and are everlasting and unchanging. "Beyond the limits of the stars", says Plato, "exist pure ideas, without shape or colour, intangible and invisible not fixed in sensible particulars but free and independent". We would very much like to include this beautiful sentence in a poem extolling a mahtematical paradise, in which dwell ideal polygons, circles, spheres, regular icosahedrons and other geometrical figures; among them all striding portentously the logarithm, surrounded by a retinue of square, cube and fourth roots... In scientific considerations such fabulous pictures seem a little odd, but we must bear in mind that in the dawn of science all theories about the universe were based not exclusively on observation and argument but were conglomerates in which the conclusions of enquiry and cool logical speculation were overlaid by poetic fancy. Centuries were to pass by before man learnt to be cautious and critical in science.

With this in mind we may appreciate more fully Plato's granting of an independent existence to ideal geometrical concepts. He imbued geometry with the character of a true science dealing with existing reality; the realistic Greek mind was apt to recoil from a science concerned merely with intellectual inventions.

If we accept the real existence of the world of geometrical forms, we must ask two questions. The first is "Why in fact to bother to examine it?" and the second "How to do it?"

The need for study and research was obvious to Plato. Only ideal geometrical forms are governed by simple laws and only they can claim to have everlasting and invariable existence. The objects of this world reproduce

these circles and straight lines only approximately and are influenced by accidents, and at the same time the relationships between them are only hazy reflections of the relationships between perfect forms. The reality we perceive thus stands as a representation of those relationships between perfect objects which play, as it were, the rôle of "pre-examples". Plato does not explain how it comes to pass, but makes his point clear by means of an allegory about prisoners in a cave, with a wall before them. At the entrance to the cave a fire is burning, and people are moving freely in front of flames. The passers-by and the objects they carry throw shadows on the wall. The prisoners may see the vague and vacillating shadows and gather from them how the objects are looking, but the shadows reproduce reality imperfectly and so the prisoners are incompletely informed.

Quite so are things in geometry. Geometrical relationships between the real objects under discussion are like shadows of the relationships between perfect beings. It is with these latter that man should try to be acquainted, for only they are primary and essential. Only they can give us true knowledge.

As for signposting the way which leads to this knowledge, we turn to the problem arising in the second of our questions. Here we discard our fabulous pictures and enter the field of the methodology of scientific work which, according to Plato, requires immense effort and indefatigable perseverance. In the scientific examination of an object Plato distinguished a number of succeeding steps. Let us take the example of a circle. The first step is the name-giving, the second the definition—that all points on a circle are equidistant from its centre. The third step consists in the producing of an image of the circle, accessible to the senses, e. g. in the drawing of it; such a picture is something "totally different from the idea of a circle". The fourth step is the scientific recognition, the embracing of the object by reason, the acquisition of an objectively true notion of it. This is achieved in a mental process, not appealing to the sense-images and without recourse to language. It is this activity of the mind which brings us closest to the essence (the idea) of the object. He who does not pass through the above four stages will never reach this essence.

Protagoras' sophistical criticism and Plato's dreams about the real world of ideas led to distinct progress, the significance of which must not be underestimated.

First of all it has now been acknowledged once and for all that geometrical concepts are not the same as their real counterparts in the material world; thus an empirical observation of a certain phenomenon is not and cannot be sufficient foundation for its acknowledgment as definite "pure geometrical" truth.

Almost without exception records of the older Greek mathematics have not survived to the present day, but we may guess that its arguments were a mélange of strictly logical deductions with references to facts which may have seemed obvious but which had not yet been precisely examined. Such a guess would be supported by the only

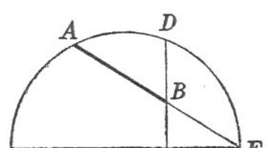

FIG. 5

longer fragment from the fifth century B.C. which has come down to us, namely, a tract on half-moons by Hippocrates. In it we find a series of precise arguments, and among them a reference without comment to the following problem: to construct a segment AB of given length with ends on a given semicircle and straight line CD respectively (Fig. 5), which when continued passes

through the given point E on the diameter of the semicircle.

It seems plausible that the possibility of constructing such a chord was assumed intuitively by Hippocrates; the given segment AB, when A moves round the semicircle and B slides on CD, will assume a position where the continuation of AB passes through E.

It is possible that not only Hippocrates concerned himself with this problem, but it is unlikely that it was solved otherwise than by trial and error. For two hundred years later we find Apollonius of Perga also treating it, which would not have been the case if the solution had already been known for centuries.

In the Platonic school such a mixture of logical argument and appeal to intuition would not do; not so much, perhaps, because of the greater logical demands as on account of the cardinal principle: that the world of empirical reality is something lower, something baser, and cannot bear upon the higher, sublime world of perfect geometrical forms.

Platonic conceptions have produced an effect upon the development of deductive thought in yet another respect: to be sure, the final aim of scientific knowledge is, according to them, the comprehension of the idea of an object, but the preliminary condition for this is the formulation of its definition. Therefore the search for the correct definition became the daily preoccupation of the Platonic school, and Plato himself devoted much effort to this in his later dialogues. There has survived in historical literature (in the *Lives of the Philosophers* by Diogenes Laertius) the following characteristic story. Man, as he used to be defined by the Platonic school, was a mortal being, two-legged and featherless. This definition became quite popular in fourth-century Athens. The philosopher Diogenes, the *enfant terrible* of the Athenian community, plucked a live cock and took it along to the Platonic school, known as the Academy, saying "Here is your man". Diogenes' objection was not dismissed as mere facetiousness but accepted in all seriousness. In fact the plucked cock was, by definition, a man. As the result of this the definition was extended by addition of the words: "and having smooth nails".

In this curious and unexpected way the metaphysical phantasies of Plato contributed a great deal to the analysis of logical argument and led to an increasing interest in mathematics and logic. The Platonic Academy produced many eminent mathematicians. One of them, Theudios, wrote a text-book of geometry. In the Academy, also, logic was first formulated as a distinct branch of science, later systematised by the Academy's most distinguished pupil, Aristotle.

§ 3. Aristotle

Aristotle was of a cool, penetrating mind, extremely erudite—the first "scholar" in the modern sense of the word—critical and not given to flights of poetic fancy. So it is no wonder that he rejected the theory of the real existence of "ideas" and wrote bluntly: "To say that ideas are patterns of things and that things contain within themselves something of ideas is idle talk." He devoted many pages of his books to his fight against the Platonic concepts which, one gathers, were widely accepted.

Having rejected the real existence of ideal straight lines and circles Aristotle was forced to face the question: "What, then, are the objects with which mathematics deals?"

Quite certainly, they are not simply the objects of this world, for "none of them is of the sort that mathematics is interested in". Every kind of knowledge deals with objects that are perceptible to the senses, which are studied in physics. In this case, says Aristotle, "we must consider wherein the mathematician differs from the physicist.

For physical bodies contain planes, solids, lengths and points—which are what the mathematician investigates... The mathematician studies these figures, not qua limits of a natural body... He separates them since they can be separated in thought." "He investigates things after eliminating all sensible qualities such as weight, lightness, hardness and softness, also heat and cold... leaving only the quantitative and continuous... He investigates them in relation to nothing else."

Thus the objects of geometry, straight lines, circles, etc., have in Aristotle's philosophy, lost the real existence given them by Plato and have become the products of a complicated process of thought. Aristotle puts it concisely: "Mathematical objects we consider as the results of abstraction; physical objects have further properties".

Aristotle describes somewhat briefly how general concepts arise, by storing in the memory the features of objects which are similar to one another, i. e. how the process of this abstraction proceeds. He is concerned chiefly with its results. The human mind carries out this process of abstraction by collecting together the simple, general characteristic features of the real observed objects. The resulting concepts give the "essence" of things and geometry gives a picture of reality, one-sided but correct. Mathematical objects, to be sure, have no separate existence from the real ones, continues Aristotle, but "we treat them as separate". Thus, in spite of the utter difference in their views, the disciples of Aristotle were talking the same language as the pupils of Plato. Even today we have not swerved from their course; we say that Napier has "discovered logarithms" in the same way that we say an entomologist has "discovered" a new and beautiful species of butterfly.

Aristotle's studies of the structure of "proving" (or as we should say nowadays, "deductive") knowledge were of the greatest significance for geometry. Analysing the process of argument—basing it, it seems, to a great extent on a contemporary text-book of geometry—he became aware that not everything can be proved, since each argument must rest on previous information, which may in turn be based on yet earlier evidence. But since this process cannot be carried on *ad infinitum*, it is necessary to draw a line somewhere. Thus knowledge must be founded upon some principles which are taken for granted without proof. These are of two kinds.

One kind expresses certain general laws which find application in many sciences, e. g. that the differences of equals are equal (a favourite example of Aristotle), which works as well in geometry as in arithmetic. Aristotle lays stress on the importance of axioms, especially those relating to the concept of quantity (another example: that two quantities equal to a third one are equal to each other). We do not know if the credit for realising the need for the precise formulation of such obvious laws must go to Aristotle or to his predecessors. In any case, these axioms have now passed into all text-books of geometry and have been acknowledged as a foundation of mathematics.

Other principles of thought, are, to Aristotle, definitions in which the human mind describes concepts which are to reflect the reality. Such a definition need not be, as Plato would have it, a stage of a mental process whose culmination is the inward grasping of the idea of an object, but it must formulate the "essence" of the object under consideration. This essence must not be understood metaphysically, but actually and descriptively. They are, argues Aristotle in his *Analytics*, those attributes of an object which are peculiar to it but whose whole cannot be predicated of any other object; this whole of necessity constitutes the essence of the object. This idea is illustrated by the example that man is a mortal being, two-legged and featherless. These features are peculiar to all individually-existing men, but not to other creatures.

Definitions are the foundations of knowledge. If we omit nothing which is peculiar to a certain object, says Aristotle, we can prove anything about it which is capable of being proved.

Nowadays we attach perhaps somewhat less importance to definitions than did Aristotle, but must admire the acuity of his further observations in which he points out that it does not suffice to formulate the definition of an object, but that one must also prove the existence of the object which is being defined. What is the point of saying what a *tragelaphus* (¹) is, if the beast does not exist? Let us explain his idea by the following example:

It is not sufficient to define parallel lines as non-intersecting straight lines lying in the same plane, but one must also prove their existence. This we do, as it must

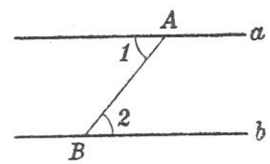

FIG. 6

have been done by the Greeks in the fourth century B.C., by constructing the lines a and b (Fig. 6) forming equal alternate angles *1* and *2* with a third line AB, and then proving that they cannot have any point in common.

There are also, according to Aristotle, certain concepts whose existence cannot be proved, since each stage of reasoning refers to a knowledge of objects whose existence has already been grasped, and continuing in this direction one comes to a stage where there is nothing on which to find support. The existence of such an object—Aristotle gives unit and magnitude as examples—should be taken for granted (postulated) and the post-Aristotelean geometry lays down explicitly the "postulates" which are necessary,

(¹) A mythical creature, half deer and half panther.

e. g. that there "exist" circles with arbitrary centres and arbitrary radii.

Amateur mathematicians would consider such postulates to be pointless, and ask flippantly how a circle could not exist, when we can draw it with a pair of compasses. Our postulate, however, formulates abstractedly precisely that it is possible to move one leg of the compasses round the other.

The statement that every science rests on principles which are "unprovable" and originate from empirical observations has effectively opposed *a-priori-ism.* The requirement that every definition be accompanied by the proof of the existence of the object defined has been accepted by science once and for all. The concept of the postulate derived therefrom has played a fundamental rôle in the further development of mathematics.

Nevertheless, not everything Aristotle tells satisfies us. He divides knowledge into definitions and proofs (as we should say, definitions and theorems). Yet what is known by the first and what by the other? Aristotle attempted to draw a line between the spheres of action of proofs and definitions, but without convincing results. As he says, "definitions are principles of proofs", and the human mind forms these as a certain extract of reality. Further details as to how in fact to make this extract of reality are missing. In particular, the question whether definitions and postulates of existence contain all that geometer does extract from his observation of the world, and whether these, together with general axioms, suffice for the establishment of this science, is not answered.

It seems that Aristotle was of this opinion.

§ 4. Euclid and the axiom on parallels

Thirty or forty years after the work of Aristotle, i. e. about 300 B.C., were written *The Elements* of Euclid, an incomparable masterpiece of systematic, deductive

Greek thought, giving in thirteen books the geometrical and arithmetical knowledge of the times. We cannot here evaluate the immense influence exerted by *The Elements* on the development of science as a whole and not only in the field of mathematics, nor can we analyse their contents. It is sufficient to say here that the geometrical books of *The Elements* coincide almost exactly with the usual school course of geometry, and Bertrand Russel tells that when he was young, Euclid was the sole acknowledged text-book of geometry for boys in Britain. For the time being we are interested only in Book I, and especially in the paragraphs dealing with the foundations of geometry.

The exceptional precision of Euclid's thoroughly logical mind enabled him to realize the fundamental fact which escaped the notice of Aristotle. The action of defining and reaching the "essence" of concepts, together with general axioms—generally speaking those which refer to the properties of quantities—does not suffice for a logically correct development of geometry. One must, in addition, accept without proof certain laws, certain specifically geometrical axioms. Aristotle's postulates of existence are such axioms, but it turned out that in geometry it is necessary to refer not only to the definitions of concepts but to certain relationships between them. To us, who are accustomed to the study of relationships between objects, there is nothing surprising in this, but to Aristotle knowledge consisted first and foremost of the study of the characteristic attributes of these objects.

We shall not quote all the axioms listed by Euclid; we are now aware that his list is not complete, that is, that he did not formulate all the necessary axioms, although he noted the most essential ones. Here, however, are two:

1. *A straight line may be drawn through two points.*

2. *If a straight line c intersects two other straight lines a and b and makes with them two interior angles on the same*

side (1 and 2 in Fig. 7) *whose sum is less than two right angles, then a and b meet on that side of c on which the angles lie.*

In Fig. 7 the lines a and b intersect on the right hand side of the line c.

Why Euclid needed this axiom, known as the *axiom of Euclid*, we shall see below; its relation to experimental data will be discussed later on.

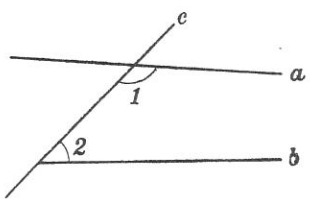

FIG. 7

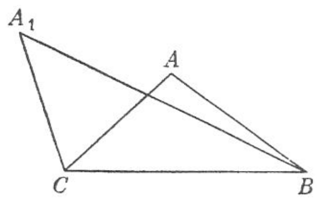

FIG. 8

The first twenty-eight paragraphs of *The Elements* develop the theorems on congruent triangles, on the isosceles triangle, on the construction of perpendiculars. We also find here the theorem that the exterior angle of a triangle is greater than either of the interior and opposite angles, and some other properties of triangles, for instance that the sum of two sides of a triangle is greater than the third side. These paragraphs also state the theorem that if we do not alter the lengths of two sides of a triangle but increase the angle between them, the third side will become longer: i. e., in Fig. 8

$$A_1B > AB.$$

All these theorems are proved without referring to the axiom of Euclid. They are, as we say, independent of it.

Euclid was quite conscious of this independence, as is obvious from the arrangement of his exposition.

In the paragraph 27 Euclid demonstrates the method of constructing straight lines with no points in common, i. e. parallel lines.

To do this it is necessary, as pointed out above, to construct on AB (Fig. 9) equal angles *1* and *2*.

The straight lines a and b forming equal interior alternate angles *1* and *2* with AB are parallel. In fact, says Euclid, if they intersect, for instance, at the point K on the right of the figure, angle *2* would be an exterior angle of the triangle ABK but would be equal to the interior angle *1*, which is impossible according to the theorem mentioned above.

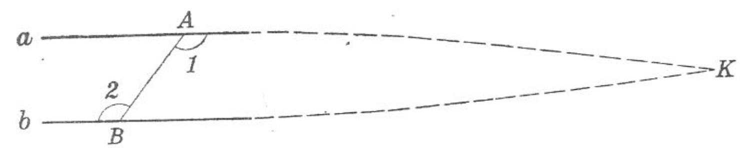

FIG. 9

As we see, this argument is not based on the axiom of Euclid. This axiom is not necessary until the beginning of paragraph 29, in which we find the proof of the following theorem: *If two straight lines are parallel, then their transversal forms with them equal interior alternate angles*, i. e. in Fig. 10,

$$\textit{if } a\|b, \textit{ then } \sphericalangle 1 = \sphericalangle 2.$$

Euclid argues as follows: $\sphericalangle 1 + \sphericalangle 3 = 180°$, since *1* and *3* are adjacent. If $\sphericalangle 1$ were greater than $\sphericalangle 2$, the sum

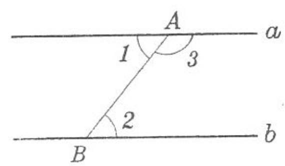

FIG. 10

of $\sphericalangle 2$ and $\sphericalangle 3$ would be less than 180°, whence by the axiom a and b would intersect. But since they are assumed to be parallel, we have the contradiction; *1* cannot be greater than *2*. Similarly it cannot be less than *2*, so the two angles must be equal.

So we see that the axiom of Euclid was necessary to prove that interior alternate angles formed by two parallel lines with their transversal must be equal.

This theorem is basic for proving many other theorems of geometry. Let us mention one or two of them.

1. *The sum of the angles of a triangle is equal to two right angles* (180°).

We construct a line e parallel to the base of the triangle through its apex A (Fig. 11).

Then $\measuredangle 4 = \measuredangle 1$, since they are equal alternate angles formed by e and BC with AB. Similarly, $\measuredangle 2 = \measuredangle 5$. But

$$\measuredangle 4 + \measuredangle 3 + \measuredangle 5 = 180°,$$

$$\measuredangle 1 + \measuredangle 3 + \measuredangle 2 = 180°.$$

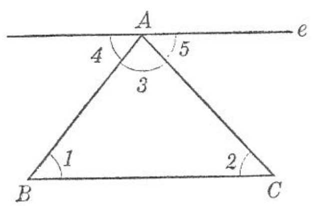

FIG. 11

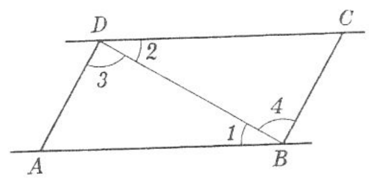

FIG. 12

2. *Through a point A not lying on the straight line b there passes one and only one straight line parallel to b.*

3. *Parallel segments contained between two parallel straight ines are equal* (Fig. 12).

We get the equality $AD = BC$ by showing that the triangles ADB and BDC are congruent; this follows precisely because the alternate angles *1* and *2* are equal (the same applies to *3* and *4*).

4. A corollary from the above theorem is that *if parallel lines intersect two straight lines a and b and cut off from one of them equal segments, then the segments cut off from the other line will also be equal*, i. e. in Fig. 13,

if segment I = sgm II, then sgm III = sgm IV.

This theorem is the basis for the so-called theorem of Thales, which runs as follows: *segments formed in a straight line a by a number of parallel lines are proportional to those formed in b*, i. e.

sgm I : sgm II = sgm III : sgm IV.

The entire theory of similar triangles is based on this theorem, since its starting-point is the figure obtained by

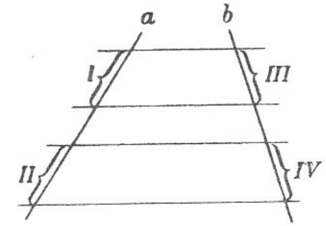

FIG. 13

intersecting the triangle with a parallel to one of its sides. Consequently, all the relationships in a triangle and all plane trigonometry follow from the theorem of Thales.

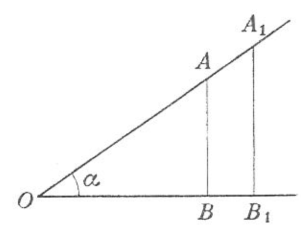

FIG. 14

Which theorem is the most essential in ordinary trigonometry? Obviously, that one which enables us to refer to the ratio of the perpendicular AB to the hypotenuse OA (Fig. 14) as *the sine of the angle α*, and to the ratio of the base OB to the hypotenuse OA as *the cosine of the angle* α.

The reader, perhaps, will protest and say: "Why should we not call the ratio "$\frac{AB}{OA}$" sine if we want to? After all, it depends only on us". Yet this is not so. When we introduce the term "the sine of the angle α" we are presupposing thereby that the ratio depends only upon the magnitude of α and does not depend on which point of the line OA has been chosen as A; in other words we have assumed that

$$\frac{AB}{OA} = \frac{A_1B_1}{OA_1}.$$

This equality follows from the theorem on similar triangles, and in turn from the theorem of Thales and finally from the axiom of Euclid.

5. The theory of the circle is also dependent upon the axiom of Euclid on that part of it which deals with the angles at the circumference and at the centre.

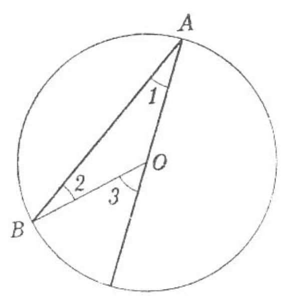

FIG. 15

In fact, to show that the angle at the circumference is half of the angle at the centre, when both have the same arc as their base, we consider the isosceles triangle AOB (Fig. 15).

The angles *1* and *2* are equal, and their sum is $180° - \sphericalangle AOB$ by the theorem on the sum of the angles of a triangle, that is, directly from the axiom of Euclid. But $180° - \sphericalangle AOB = \sphericalangle 3$, so $\sphericalangle 1 + \sphericalangle 2 = \sphericalangle 3$ and $\sphericalangle 1$ is half of $\sphericalangle 3$.

Therefore the theorem on the geometrical locus of a point from which a given segment is visible at a right angle also follows from the axiom of Euclid.

6. Let us finally mention the theorem that *parallel lines are equidistant*, i. e. in Fig. 16, that

If the straight lines a and b do not intersect, then the distances of all points on a from the other line are equal.

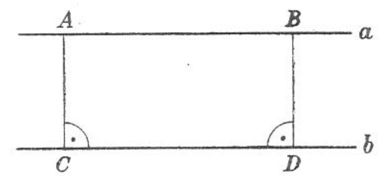

FIG. 16

Such distances are the lines AC and BD perpendicular to b. They form equal alternate angles with b, whence they are parallel.

We now have parallel segments between parallel straight lines and it follows by the theorem quoted at No. 3 above, thus indirectly by the axiom of Euclid, that $AC = BD$.

Conversely, points equidistant from b and on one side of it form a straight line parallel to b.

The above review demonstrates how great is the rôle played by the axiom of Euclid in geometry—how little would be left if we no longer accepted the truth of this axiom in our school-books. Naturally, Euclid was not the first to use it; the fact that a transversal intersecting two parallel lines forms equal interior alternate angles was, in all probability, known for a very long time before. It is unlikely that Eupalinus, without it, would have succeeded in digging his tunnel on Samos. In any case, the axiom of Euclid is by no means a deeply-hidden truth. Direct observation shows that if two actual straight lines a and b form the angles *1* and *2* with the line AB (*2* in Fig. 17 is a right angle) and the sum of $\sphericalangle 1$ and $\sphericalangle 2$ differs from 180°, then a and b will intersect. It is easy to discover by experiment that if this sum were 178° the distance of the point of intersection of the lines from B, in Fig. 17, would be about three yards. This experiment would provide a little more difficulty, but would still be practicable, if the sum of *1* and *2* were 179°50′, since the distance would then be 34 yards.

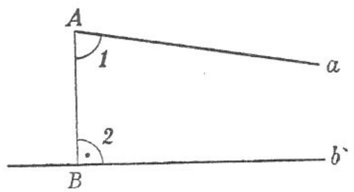

Fig. 17

The properties of parallel lines, of parallelograms etc., were known in the Pythagorean school and there is reason to believe that the proofs of many theorems, such as that on the sum of the angles of a triangle, were the same as they are now. Nevertheless, the theory was not entirely satisfactory. This we know from the writings of Aristotle, namely from his *Analytics*. There he criticises the procedure of those who would prove a certain property (A) on the basis of (B), which in turn is derived from (C), and finally conclude (C) from (A). Thus according to Aristotle, they assert that something holds because it holds. He reproaches the contemporary theory of parallels with this logical fallacy, this vicious circle. To Aristotle's pupils, for whom the *Analytics* were written, the above comments were clear, but we can only surmise how geometry used to be exposed in the fourth century B.C. Greek literature

of later times indicated on several occasions the insufficiency of the following argument, which, as one may infer from the criticism, used to be applied:

Let the angle *2* in Fig. 17 be right and the angle *1* acute. Then, as is obvious and not difficult to prove, the line a will approach line b, if we move along a on the right hand side of A. Hence, a must finally intersect b.

It may be further deduced that the axiom of Euclid still holds in the case where *2* is not a right angle, but the sum of the angles *1* and *2* is less than 180°.

The above argument, stating that two lines approaching each other will necessarily intersect, must have been rejected the moment it was discovered that lines may approach nearer and nearer and yet not intersect. Such lines, for example, are the hyperbola with equation $y = 1/x$, which was already known to the Greeks as a section of a cone, and its asymptote, the horizontal axis (Fig. 18).

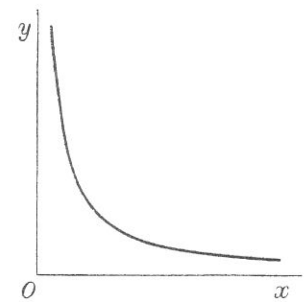

Fig. 18

The hyperbola comes infinitely close to the horizontal axis, but does not intersect it.

Thus the unlimited approach of two lines to each other by no means implies that they will intersect. In order to prove that the straight lines a and b will intersect (Fig. 17), it is not sufficient to show that they approach each other, but one must also appeal to other properties which are peculiar to straight lines but not to others, for example to hyperbolas.

Many readers will hasten to protest that the matter is clear by itself, "immediately evident", that straight lines cannot approach and not intersect. Thus saying, they arm themselves with their conviction of the truth of this theorem from intuition, that is from their experience of geometry; they assume the theorem as a self-evident truth, not deduced from other theorems. Which means that they accept the theorem as an axiom—just as Euclid did.

Aristotle's allusion probably did not apply to the fallacy mentioned above, since there is no vicious circle inherent in it. We may, however, find a vicious circle in the following argument, which is a simplified version of the reasoning of Geminus, a mathematician living some centuries after Euclid. It has been passed down to us by an Arab commentator.

It is easily seen that the axiom of Euclid (let us call it (A) in the terminology of Aristotle's criticism on page

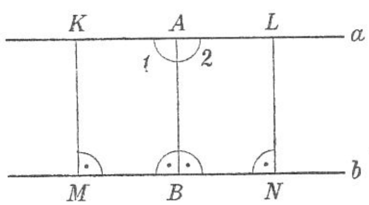

FIG. 19

32) follows from the theorem (B), that if a straight line is parallel to one of two parallels a and b it will also be parallel to the other. We shall deduce (B) from the theorem (C), that the straight line a is everywhere equidistant from the straight line b, or in other words that all segments with one end lying on a and perpendicular to b are equal.

Let us take $BM = BN$ and let KM, AB, LN be perpendicular to b (Fig. 19). The quadrangles $ABMK$ and $ABNL$ are congruent, since BN and BM are symmetrical about AB, whence KM and LN as perpendicular to them and equal (by (C)) are also symmetrical about AB. Hence, the corresponding angles 1 and 2 of both quadrangles are equal. These are adjacent, therefore right.

So far, so good, but we must now justify (C). True, we have done this on page 31 (the theorem at No. 6), but we were then basing our argument on the equality of interior alternate angles, and consequently on the axiom of Euclid (theorem (A)). The vicious circle has now been closed in conformity with the criticism of Aristotle: the truth of (A) has been made dependent on the proof of the truth of (A). For the sake of historical accuracy we should mention that Geminus did not base his argument on the theorem (C), stating that the points lying on a straight line a parallel to a straight line b are equidistant from b, but on its converse, that points equidistant from b and lying on the same side of it form a straight line not cutting b. The essential factor is, of course, that all such points form a straight line, and not some curve. Geminus, and quite a number of his followers, tacitly assumed this, that is, they considered it self-evident, a necessary property of straight lines, and by no means a new axiom.

We note in passing that points equidistant from a certain line l do not always form a line of the same shape as l. In Fig. 20 is drawn a parabola and a number of points equidistant from it; these points quite obviously do not lie on another parabola.

The fact that points equidistant from a straight line form another straight line is a characteristic property and we might repeat the remarks made above when discussing lines which approach but do not meet.

This investigation has given a positive result: namely, that the axiom of Euclid and the theorem according to which points equidistant from a straight line and on the same side of it lie on another straight line are equivalent statements in the sense that one of them can be deduced if the other is accepted as an axiom. Geminus did not realize this, and imagined that his exposition of geometry dispensed with the axiom of Euclid.

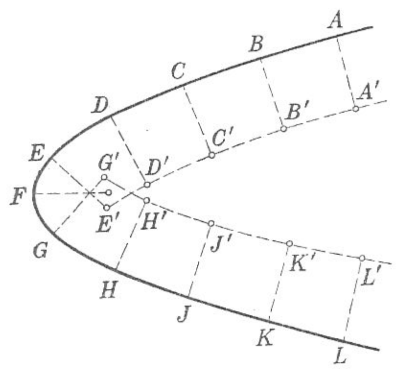

FIG. 20

Examining this fallacy of Geminus—our Arab commentator mentioned above repeats the former's arguments with applause—we must admire all the more the critical acuity of Euclid, who was not led astray by what appeared to be a proof and who considered it necessary to assume an axiom—rightly bearing his name.

Euclid was fully aware of the significance of his standpoint: the theorems of Book I of *The Elements* were arranged, as we have mentioned, in order to delay as long as possible the introduction and application of the axiom, even though its earlier use would have simplified some proofs. His aim was methodical—to lay down all that could be proved without appealing to the axiom on parallels, or in other words, to segregate that part of geometry which is independent of this axiom. This part of geometry is nowadays sometimes referred to as *absolute geometry*—a rather odd term, originating from Bolyai. Euclid used no such term, but realized to some extent

the importance of dividing absolute theorems from the rest. One should not gather from this that he considered the latter to be less reliable than the former; he was, rather, governed by a logician's instinct and pigeonholed together those theories which rested on common or analogical foundations. These tendencies appear frequently in modern mathematics, especially in algebra, and do give a certain tone to the science.

As we have said, it was logical requirements which decided Euclid to assume the axiom on parallels, he thought this axiom be necessary for the correct development of geometry. This point of view had several consequences; apart from anything else it was an opposition to Aristotle's methodological directions, for Aristotle wished to found geometry on general axioms such as "two quantities equal to a third are equal to each other" or "the part is smaller than the whole" and on definitions which fixed the meaning of geometrical concepts. The results of observing the outward world of geometrical forms are expressed, according to Aristotle, in the choice of suitable abstract notions, copies, so to say, of reality in thought. The postulates of existence form the bridge to connect these notions with reality. Geometrical theorems are deduced systematically and consecutively from the essence of these concepts formulated in definitions. Putting it concisely and somewhat simplifying, Aristotle wished to build up the science of space on the general laws of human thought and on the definitions. His idea is not missing in modern science; many mathematical theories are developed in this way, this procedure is used, for instance, in four-dimensional geometry.

Having introduced his axiom Euclid did, to a certain extent, break away from the narrow Aristotelean pattern. He found, and future ages fully agreed with him, that there was nothing in the essence of the concept of a straight line which would force one to assume that two straight lines forming with a third unilateral interior angles with a sum less than 180° must necessarily intersect. Hence, this property should be formulated separately. Therefore it is from Euclid that we have the view that geometry is to be founded on concepts and axioms relating to them (that is, not general axioms, but specifically geometrical axioms)—a view wider than that held by Aristotle. The process of arriving at the knowledge of the real world lies not only in a skilled formulation of concepts but also in the fixing of the principal relationships between them. The theorems of geometry are then deduced from the axioms without a repeated appeal to intuition.

The above views do need any special comment, since nowadays they are generally acknowledged and taught in schools. Nevertheless it would be worth while to mention that over the course of centuries they met with lack of comprehension and with opposition. In 1733 still the Italian mathematician G. Saccheri tried to prove, in a dissertation entitled *Euclides ab omni naevo vindicatus* (1), that two straight lines cannot intersect in two points and also that there is a straight line which passes through two given points. These were not, of course, proofs in the present meaning of the word but somewhat primitive appeals to intuition in which Saccheri considered an arbitrary curve between points A and B (Fig. 21), rotating it from the left-hand side of the points A and B to the right-hand side, and finally bringing both curves closer to each other until they coincided. Saccheri did not grasp the depth of Euclid's conceptions but was a shrewd man widely gifted. He used to play simultaneously three games of chess by memory, that is without looking at the board—and successfully too. In the dissertation quoted he has a proof of the axiom of Euclid (two proofs, in fact, but both wrong) which was, to him, the chief flaw in the Elements, and in doing so he noted and correctly proved an interesting theorem from absolute geometry.

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(1) "Euclid cleared of all stain".

We mentioned Saccheri's attempts not because they have played any great rôle in the history of mathematics but in order once more to pay homage to the sharpness

FIG. 21

of Euclid's intellect. Not only fortune, but also opinion is variant. What Saccheri considered a fault in Euclid's exposition we now take to be one of his chief merits.

§ 5. Attempts to prove the axiom of Euclid

Saccheri's standpoint, as presented in the preceding section, was extreme. The necessity for taking not only general axioms as a base for geometry but also some specifically geometrical ones was by and large acknowledged, and axioms like "there exists a straight line which passes through two given points" caused no objections. They were not, after all, a far cry from the Aristotelean postulates of existence. Matters were different with the axiom on parallels. Almost from the very moment when *The Elements* appeared until the nineteenth century (over two thousand years!) this axiom continually aroused opposition and many attempts were made to rid geometry of it. There is something deeply moving in the epic of these heroic strivings towards ideal scientific perfection—disinterested effort directed solely by the love of knowledge.

In these endeavours it is possible to distinguish two main trends, which crossed, however, quite frequently.

First, it was attempted to replace the axiom of Euclid by another one equivalent to it. Such an axiom is, as we have seen, the one "that all points equidistant from a straight line and lying on the same side of it also lie on a straight line".

In the seventeenth century the Englishman Wallis showed that the axiom of Euclid is equivalent to the existence of triangles of different sizes but with equal corresponding angles—i. e. to the possibility of magnifying the triangle without changing its angles. In other words, if the axiom of Euclid were not true similar triangles could not exist, or, more generally, similar figures altogether, and we would be unable to scale plane objects, i. e. make maps. Wallis considered that an axiom which states the existence of similar figures lays down a more characteristic geometrical property than does the axiom of Euclid, whence it should be given precedence. Finally, it was discovered in the eighteenth century (Saccheri and others) that *the axiom of Euclid is equivalent to the theorem on the sum of the angles of a triangle.* As we have

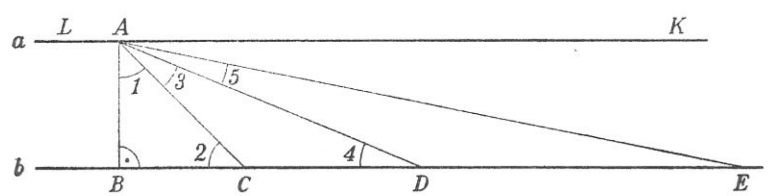

FIG. 22

seen, this theorem follows from the axiom of Euclid (the theorem at No. 1, page 28). Conversely, if we suppose that the sum of the angles in an arbitrary triangle is 180°, we can easily demonstrate the truth of the axiom of Euclid.

According to the comment made on page 34 it is sufficient to show that if the straight lines a and b are parallel (Fig. 22) and AB is perpendicular to b, then AB will be perpendicular to a.

Let us take $BC = AB$. $\triangle ABC$ is a right-angled triangle, the sum of its acute angles is $90^\circ = \mathrm{D}$ (we infer it from the theorem that the sum of the angles of a triangle is 180°). Since these acute angles are equal, we have

$$\sphericalangle 1 = \tfrac{1}{2}\mathrm{D}.$$

Let us now take $CD = AC$. The angle $ACD = 2\mathrm{D} - \sphericalangle 2 = 2\mathrm{D} - \frac{1}{2}\mathrm{D}$, whence the sum of $\sphericalangle 3$ and $\sphericalangle 4$ equals $\frac{1}{2}\mathrm{D}$, and these angles are equal, giving

$$\sphericalangle 3 = \tfrac{1}{4}\mathrm{D}.$$

Let $DE = AD$. As before we obtain

$$\sphericalangle 5 = \tfrac{1}{8}\mathrm{D}.$$

Continuing the process we have at the vertex A the following angles:

$$\tfrac{1}{2}\mathrm{D}, \tfrac{1}{4}\mathrm{D}, \tfrac{1}{8}\mathrm{D}, \tfrac{1}{16}\mathrm{D}, \ldots$$

The angle KAB is greater than $\sphericalangle 1$, than $\sphericalangle 1 + \sphericalangle 3$, than $\sphericalangle 1 + \sphericalangle 3 + \sphericalangle 5$, etc., i. e. it is greater than each term of the sequence

$$\tfrac{1}{2}\mathrm{D}, \tfrac{3}{4}\mathrm{D}, \tfrac{7}{8}\mathrm{D}, \tfrac{15}{16}\mathrm{D}, \ldots$$

which tends to D. So the angle KAB is equal to D or is greater than D:

$$\sphericalangle KAB \geqslant \mathrm{D}.$$

The same argument gives

$$\sphericalangle LAB \geqslant \mathrm{D}.$$

The sum of the angles KAB and LAB is 2D, whence owing to the above inequalities they are both right angles.

Thus the equivalence of the theorem on the sum of the angles of a triangle to the theorem of Euclid has been proved.

Secondly, it was attempted to deduce the axiom of Euclid from the remaining axioms. Several arguments which claimed to do this made use of facts which were obvious to their authors but were, when thoroughly examined, equivalent to the axiom of Euclid—that is, facts which could not have been proved themselves without the aid of that axiom. These arguments, then, contained the same logical fallacy as that of Geminus. Some of these "proofs" were quite ingenious and interesting. As an example we shall give here the proofs of the French mathematician Legendre, who inserted them into a school text-book which was in general use in the first decades of the nineteenth century. The poor schoolchildren were hammering away at false arguments until someone noticed the mistakes and when Legendre removed them from his book. But he still did not give up and found some more false proofs which were published as a special paper in 1833!

In the proof which we shall quote here Legendre tried to show that if we suppose the sum of the angles of a triangle to be greater than 180°, or to be less than 180°, we are faced with a contradiction.

1. Let us suppose that the sum of the angles of the triangle ABC (Fig. 23) equals $180°+\alpha$.

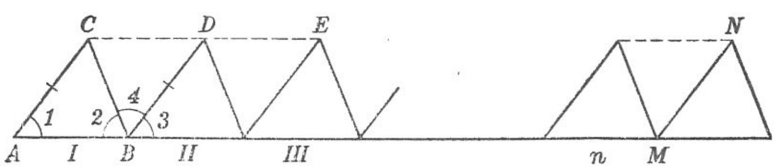

FIG. 23

We now construct on the same straight line AB a series of triangles congruent to ABC, as in the figure. Let us now join their apices. We then have

$$\sphericalangle 1 = \sphericalangle 3$$

but

$$\sphericalangle 1+\sphericalangle 2+\sphericalangle C = 180°+\alpha \quad \text{(by assumption)},$$

whence

$$\sphericalangle 3+\sphericalangle 2+\sphericalangle C = 180°+\alpha.$$

On the other hand

$$\sphericalangle 3+\sphericalangle 2+\sphericalangle 4 = 180°.$$

Comparing this formula with the preceding one we get

$$\sphericalangle 4 < \sphericalangle C.$$

The ticked sides of the triangles ABC and CBD are equal, and side BC is common. The angle C in the first triangle is greater than the corresponding angle 4 in the second. Whence, from the absolute theorem mentioned on page 26 (Fig. 8), we obtain

$$AB > CD,$$

that is to say,

$$AB-CD > 0.$$

Let us now take a sufficiently large integer n so that $n(AB-CD)$ will be greater than $2.AC$, i. e.

$$n.AB-n.CD > AC+AC.$$

Let us consider (Fig. 23) n consecutive triangles like ABC and n more like BCD. MN is the side of the last one.

Since $n.AB = AM$ and $n.CD$ is the length of the polygonal line between C and N, namely the line $CDE\ldots N$, we may write the last inequality as follows:

$$AM-CDE\ldots N > AC+MN \quad \text{(since } AC = MN\text{)}$$

or

$$AM > AC+CDE\ldots N+MN.$$

On the left-hand side we have the segment AM, on the right the length of the polygonal line between A and M. Thus the straight distance from A to M would be longer than the roundabout distance, which is impossible.

We come to a contradiction, so the sum of the angles of a triangle can never be greater than 180°.

2. Now let us suppose that the sum of the angles of a given triangle ABC is $180° - \alpha$.

Let the angle A be acute.

Let us construct on BC a triangle BCD congruent to ABC so that the sides similarly ticked in Fig. 24 will be equal. Let us draw a straight line passing through the point D and intersecting the sides of the angle BAC at the points K and L.

Four triangles have been formed, marked in the figure by the numerals I, II, III, IV.

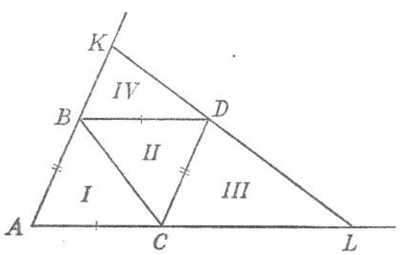

FIG. 24

The sum of the angles of each of the triangles I and II is $180° - \alpha$, and the sum of the angles of III and IV cannot exceed 180°, according to the result of the previous argument.

Let us add all the angles of I, II, III, and IV. We obtain a sum not exceeding

$$(180° - \alpha) + (180° - \alpha) + 180° + 180° = 720° - 2\alpha.$$

This sum consists of angles A, K, L and the three angles at B, C, D which are all equal to two right angles:

$$\sphericalangle A + \sphericalangle K + \sphericalangle L + 3.180°.$$

we get

$$\sphericalangle A + \sphericalangle K + \sphericalangle L + 3.180° \leqslant 720° - 2\alpha,$$

and consequently the sum of the angles of the triangle AKL does not exceed $180° - 2\alpha$.

Applying the same process to triangle AKL we get (Fig. 25) a triangle AK_1L_1 whose angles have a sum not greater than $180° - 4\alpha$. Similarly we arrive at a triangle AK_2L_2, the sum of whose angles is not greater than $180° - 8\alpha$, etc. This process leads to a triangle the sum of whose angles is negative, since 2α, 4α, 8α, 16α, ... must eventually exceed 180°.

We have come to contradiction, says Legendre, so the sum of the angles of a triangle can never be less than 180°. Since this sum can be neither greater nor less than 180° it must be 180°. From this, as we know, follows the truth of the axiom of Euclid.

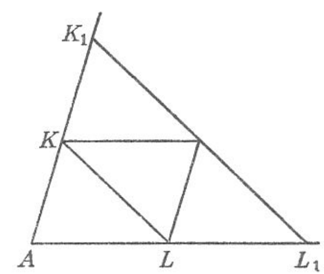

FIG. 25

The first half of the proof is correct. The theorem that the sum of the angles of a triangle cannot exceed 180° has been proved without appealing to the axiom of Euclid and therefore belongs to absolute geometry. The second half, however, is false. The reader is recommended not to read further but to try and detect the slip on his own. If he succeeds it will testify to his critical abilities and promise much for his further powers of observation and promise much for his further mathematical studies.

We will give away the secret. The weak point of the argument is the phrase: "Let us draw a straight line passing through the point D and intersecting the sides of the angle BAC at the points K and L...". It is not difficult to find a straight line passing through a point inside the angle and intersecting one of its sides—it suffices to join D with any point on this side. But how can we be so sure that a straight line exists which passes through D and intersects both sides? This should be checked. A detailed analysis based on the axiom of Euclid shows that such a line is, for instance, the perpendicular dropped from D onto the bisectrix of the angle A. Therefore it follows from the axiom of Euclid that there exists a straight line with the indicated property. The second half of Legendre's proof shows that if we can find a straight line like this through any interior point of an angle the axiom of Euclid will be true. The mist has lifted a little. It now seems, much, no doubt, to the surprise of the reader, that the axiom of Euclid is equivalent to the theorem: *Through every point within an angle there passes a straight line which intersects both sides of the angle.*

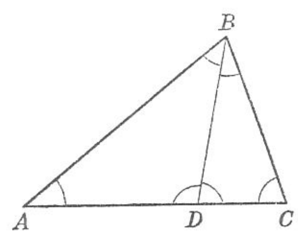

FIG. 26

The axiom of Euclid really seems like a magician with many guises, who takes us aback and deceives us.

We shall now deduce from theorem 1 a corollary enabling us to formulate still another statement equivalent to the axiom of Euclid.

Suppose that the sum of the angles in one triangle ABC is 180° (Fig. 26). Let us divide this triangle into the two triangles ABD and DBC the sums of whose angles are

denoted by α and β respectively. Hence $\alpha+\beta$ is the sum of all the angles marked with arcs in the figure and is equal to the sum of the angles of triangle ABC, i. e. 180°, plus the sum of the two angles at D. Consequently

$$\alpha+\beta = 360°.$$

As we know, neither α nor β can exceed 180°, hence $\alpha = \beta = 180°$.

Further subdivision of either triangle will again produce triangles the sum of whose angles will be 180°, and therefore: *If the sum of the angles of a certain triangle is 180°,*

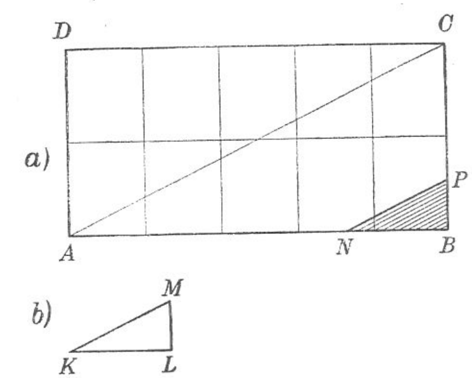

FIG. 27

then each triangle cut off from it will also have the sum of its angles equal to 180°.

We now affirm that the axiom of Euclid is equivalent to the following statement:

There exists at least one rectangle, that is a quadrangle with four right angles.

PROOF. If there exists one such rectangle, we may (Fig. 27) arrange identical ones next to each other so as to get a rectangle $ABCD$ with arbitrarily large sides. Let us consider half of $ABCD$—the triangle ABC. The sum of the angles of ABC is one half of the sum of the angles of $ABCD$, i. e. 180°.

In this case the sum of the angles of any right-angled triangle KLM is 180°, since KLM may be placed on ABC (provided that the latter has sufficiently long sides). Then, by the lemma given above, the sum of the angles of the triangle BPN is 180°, since BPN is cut off from ABC.

Now, since the sum of the angles of an arbitrary right-angled triangle is 180°, it follows from Fig. 28 that the sum of the angles of any triangle is also 180°, whence the axiom of Euclid holds.

It is clear that the converse also holds, and so the announced equivalence has been proved.

A slight modification of our argument would give the following, striking result: *If there exists at least one triangle the sum of whose angles is* 180°, *then the axiom of Euclid will be true.*

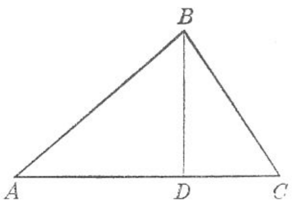

FIG. 28

In fact, if the sum of the angles of triangle ABC (Fig. 28) is 180°, then the same holds for the right-angled triangle ABD cut off from it. With two such triangles we can construct a rectangle and obtain thereby the conditions of the preceding theorem. Hence, the situation is extremely characteristic: *If the sum of the angles of at least one triangle is* 180°, *then the axiom of Euclid will hold and the sum of the angles of any other triangle will also be* 180°.

If the sum of the angles of at least one triangle is less than 180°, then the sum of the angles of any other triangle will also be less than 180°.

These theorems have been proved without appealing to the axiom of Euclid and so belong to the realm of absolute geometry.

§ 6. The axiom of Euclid and the empirical knowledge

All our considerations up till now have referred to the logical structure of geometry. Their object was to discover whether the axiom of Euclid was indispensible to the structure of geometry or whether it could be deduced from other axioms; they are, so to say, logical amusements. No single mathematician entertained any doubts as to the truth of the axiom of Euclid. This "revolutionary" idea was first conceived by the great German mathematician Gauss in the first two decades of the last century. To him the question of whether the axiom of Euclid were true was of actual, physical significance; namely, it was a matter of whether real points and straight lines, as for example, those employed in land-surveying, would obey this axiom.

The last theorem of the preceding sections yielded a method of solving the question: one should measure the sum of the angles in any one triangle. If this sum appeared to be 180°, the axiom of Euclid would hold.

Gauss traced a triangle in the neighbourhood of Göttingen whose sides were thirty or so miles long and whose vertices were at the summits of mountains.

Then with the utmost precision he measured its angles. It appeared that the deviation of their sum from 180° lay within the limits of inevitable errors of measurement, and so it remained unsettled whether this sum was exactly 180° or differed from 180° by an amount less than those errors.

We shall now show that the failure of the attempt could have been forecast in advance. To this end we shall consider some factors related to those dealt with in the last section.

Let us call the difference between the sum of the angles of the triangle and 180° the *defect* of the triangle.

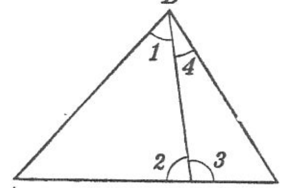

FIG. 29

Let us now divide the triangle ABC (Fig. 29) into triangles ABD and BCD. We have

$$\text{Defect } \triangle ABD = 180° - \measuredangle A - \measuredangle 1 - \measuredangle 2.$$

$$\text{Defect } \triangle BCD = 180° - \measuredangle C - \measuredangle 3 - \measuredangle 4.$$

Let us add these formulae.

$$\text{Defect } \triangle ABD + \text{Defect } \triangle BCD$$

$$= 360° - \measuredangle A - \measuredangle C - (\measuredangle 1 + \measuredangle 4) - (\measuredangle 2 + \measuredangle 3).$$

But

$$\measuredangle 2 + \measuredangle 3 = 180° \quad \text{and} \quad \measuredangle 1 + \measuredangle 4 = \measuredangle B.$$

We obtain

$$\text{Defect } \triangle ABD + \text{Defect } \triangle BCD$$

$$= 180° - \measuredangle A - \measuredangle B - \measuredangle C = \text{Defect } \triangle ABC.$$

The defect of the triangle ABC is equal to the sum of the defects of the triangles of which triangle ABC consists.

This theorem may be generalised for every partition of a triangle into triangular components; furthermore, for

every partition of a polygon into polygonal components, provided, of course, that the defect of a polygon has been defined.

Let us now imagine that the triangle whose vertices are at the Sun, Earth and Mars has a defect of 1°, i. e. its angles add up to 179°. Let us divide it into smaller ones more or less of the size of that drawn by Gauss. The distances from each other of the Sun, the Earth and Mars amount to hundreds of millions of miles and it is easy to compute that the number of the component triangles will exceed a trillion. Thus the defect of a component triangle with a thirty-mile side would be something like a trillionth of a degree. Obviously, no instrument could detect such a tiny angle. This calculation, which is based on the theorem of defects, inclines one to believe that it would not be possible to meet with noticeable defects when measuring triangles on earth. As far as the earth is concerned the defect of a triangle is virtually zero and the sum of the angles of a triangle is 180°: in other words, the axiom of Euclid with all its consequences holds. Less microscopic defects would occur only in the triangles which occur in astronomical research. The biggest triangles accurately known are those serving for the determination of the parallaxes of fixed stars.

Let G be a fixed star, A and B two opposite positions of the Earth in its orbit round the Sun (Fig. 30). The segment AB is the diameter of the Earth's orbit. Its length is about 186 million miles. The angles GAB and GBA can be measured since their vertices lie on the Earth. Of course, we may choose A and B so that the angles are equal, which will happen if AB is perpendicular to GM. The quantity $\frac{1}{2}(180° - \measuredangle GAB - \measuredangle GBA)$, i. e. $90° - \measuredangle GAB$ is called the *parallax* of the fixed star G. If the sum of the angles of the right-angled triangle GAM is 180°, then the parallax of the star will be $\measuredangle AGM$, i. e. the angle which the radius AM of the Earth's orbit would subtend at G. If the sum of the angles of the triangle AGM were less than 180°, the parallax of the star would not be equal but greater than the angle AGM.

It is evident that the defect of AGM, which is $180° - 90° - \sphericalangle GAM - \sphericalangle AGM$, is less than the parallax, which is $90° - \sphericalangle GAB$. Hence, we can estimate

$$\text{Defect } \triangle AGM < \text{parallax of the star } G.$$

The parallax of the star Sirius has been measured as 0·38″, and that of Vega as 0·08″. The corresponding defects are smaller than the parallaxes and are therefore

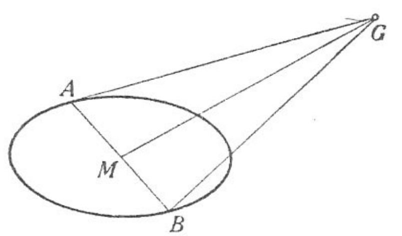

FIG. 30

very tiny angles. Only much greater triangles than those used when measuring parallax, which are very "narrow" indeed, could have larger defects; in the case of Sirius the side AG is about half a million times as long as AM.

No triangles bigger than parallax triangles can possibly be investigated and so the question of whether the defects of enormous triangles are zero, in which case the axiom of Euclid would hold throughout space, or not, has not been settled by the direct measurement of angles. Contemporary physical theories, viz. the theory of relativity, state that when we are dealing with very great distances the axiom of Euclid fails. We cannot here discuss these difficult matters nor consider the extent to which empirical data bear out the above view. In any case we must

bear in mind the possibility that the sum of the angles of a triangle, although equal to 180° with overwhelming precision when dealing with terrestrial or even solar dimensions, may be less than this figure in triangles of "cosmic" sizes. Thus mathematics finds itself faced with the task of discovering the properties of triangles on the assumption that the sum of their angles is not 180°. What would, for instance, the theorem of Pythagoras look like then? How can the area of a triangle be computed? What is the relationship between the side of a right-angled triangle and the hypotenuse and one of its acute angles?

That system of geometry which is built up on all the axioms of ordinary geometry except that of Euclid and on the negation of the latter is known as *non-Euclidean geometry* (sometimes, for reasons which we shall not give here, *hyperbolic non-Euclidean geometry*). This science contains, clearly, all the absolute theorems, which remain the same as in ordinary, Euclidean geometry, but many theorems in addition which are different from the Euclidean ones: an example, which we know, is the theorem that there is no quadrangle with four right angles.

§ 7. The creators of non-Euclidean geometry

The first scholar who realised that non-Euclidean geometry might exist and who admitted it the right to exist was Gauss. He discovered many of the theorems of the new science, but printed none of his findings. Those which we know are gleaned from his note-book, which was published in later times. It is not known whether it contains all his research and his findings. With this reservation we may conclude, on the basis of the existing materials, that Gauss, absorbed in other work, did not come to very final results in the field of non-Euclidean geometry. This is true at any rate of the methods he used, if not of the actual contents of the theorems he discovered; indeed, in the most important of the preserved fragments he uses methods of differential geometry, a science based on differential calculus, whereas synthetic methods like those used in elementary geometry would much better fit the case. This would seem, to the present author, to be one of the reasons for Gauss's not communicating his findings to the world of science.

Meanwhile two young mathematicians, the Russian, Nikolai Lobatchevsky, and the Hungarian, János Bolyai, by a bold stroke of genius, developed the principles of non-Euclidean geometry, and settled nearly all of its essential problems.

Lobatchevsky (1793-1856), a professor at Kazan University, published his first paper *On the principles of geometry* in the 1829-30 numbers of a journal which appeared in Kazan but did not reach other countries.

Bolyai (1802-60), an officer of the Austro-Hungarian army, presented his discoveries, carried out independently of those Lobatchevsky's, in a paper entitled *Appendix scientiam spatii absolute veram exhibens* (1) that appeared in 1832, few years later than the publication of Lobatchevsky's work. Thus the priority of discovery must go to the latter, and non-Euclidean (hyperbolic) geometry is accordingly called also *Lobatchevskian geometry*. Incidentally, it is truly amazing to what extent the trains of thought of the two scholars were related; they were in essence both based on the properties of the *horosphere* (*vide* § 17, p. 116).

Both dissertations were ignored by the scientific world and brought their authors none of the acknowledgement which their independence of ideas, ingenuity of argument and perfection of results deserved. They were both aware

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(1) "Appendix giving an absolutely true science about space". The term "appendix" derives from the fact that the dissertation appeared as a supplement to a text-book of mathematics written by Bolyai's father.

of the value and importance of their work. Bolyai wrote with pride to his father: "I have created a new world out of nothing". Both expected, and were entitled to expect a rightful appreciation. Both met with complete indifference or even, in the case of Lobatchevsky, with jeers from people who were somewhat narrow-minded and failed to comprehend what it was about. Lobatchevsky and Bolyai were both bitterly disappointed but reacted, however, in different ways. Bolyai, exasperated, closed his mouth and withdrew from scientific activity. Lobatchevsky took up the struggle for the triumph of his ideas; in publication after publication he doggedly justified his non-Euclidean geometry from every point of view and indicated its applications in the integral calculus in the hope that he would finally win comprehension and acknowledgement. He dictated his last work, *Pangeometria*, seriously ill, almost blind, but not giving up the struggle even in the last days of his life. The extraordinary steadfastness of spirit shown by Lobatchevsky during his twenty-five-year struggle in utter isolation has very few equals in the history of science. We must, however, admit that a university professor is able to preserve his independence more easily than a minor functionary like Bolyai.

In the sixties and seventies of the last century the concepts of non-Euclidean geometry spread and its creators, ignored during their lifetimes, were included in the Pantheon of the greatest scholars. This change of minds was certainly influenced by the publication of Gauss's correspondence, which was not sparing in its praise of Lobatchevsky and Bolyai. A more essential reason, however, was the natural evolution of scientific interest which took to questions of geometry, regarding them from new vantage-points. The celebrated works of Staudt appeared, analysing the principles of projective geometry, and Riemann's lecture *On hypotheses basic to geometry* made an immense impression.

With the air thus cleared, many scholars [1] turned to the problems of non-Euclidean geometry, filling in the still-existing gaps and refining its methods.

Of greatest significance was the lecture of Riemann. It created a very general science, known as *Riemannian geometry*, whose special, in fact very special, cases are the Euclidean and non-Euclidean geometries. This science exceeds both the scope of the present book and any possibilities of an elementary presentation, so we shall not discuss it further. Our aim is to give the principles of non-Euclidean geometry as they were formulated by Lobatchevsky and Bolyai, but with some simplifications, the most important of which are due to the recent Danish mathematician Hjelmslev [2]. We shall also give in detail—in honour of the centenary of Lobatchevsky's death in 1856—what is perhaps the most beautiful idea in his dissertation, the use of the properties of the horosphere to prove the fundamental theorem of non-Euclidean geometry. Nowadays other proofs of this theorem exist, but to our mind it is the original method of Lobatchevsky which is the most interesting.

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(1) Beltrami, Klein, Lie *et al.*

(2) Theorem and transformation *j* of § 9.

CHAPTER II

THE PRINCIPLES OF NON-EUCLIDEAN GEOMETRY

§ 8. Fundamental assumptions

Lobatchevskian geometry, as we said in the previous chapter, differs from ordinary geometry only in so far as it rejects the axiom of Euclid, so that all the theorems of Euclidean, ordinary geometry which do not rest on this axiom, i. e. the absolute theorems, are equally valid in Lobatchevskian geometry. These are the theorems which are discussed in school text-books before the chapter about parallel lines—namely, those about congruency, symmetry rotations together with their various consequences, as mentioned in part on p. 26. Also, that part of the knowledge of the properties of the circle which deals with chords and their corresponding arcs, with tangents and intersections of circles is also "absolute" (for instance, the greater the chord the nearer it is to the centre of the circle). Then there are numerous stereometric theorems which do not depend on the axiom of Euclid. They include all those which refer to the perpendicularity of straight lines and planes, for example:

1. *All straight lines perpendicular at one given point to a given line lie in one plane.*

2. *A plane which passes through a straight line perpendicular to another plane α will also be perpendicular to α.*

3. The so-called theorem on three perpendiculars: *If a line a is perpendicular to another line b lying in plane α then the projection of a onto α will also be perpendicular to b.*

All the theorems of elementary school geometry whose proofs appeal neither directly nor indirectly to the