UNM-PNM STATEWIDE MATHEMATICS CONTEST XXXVIII

November 2005 FIRST ROUND SOLUTIONS

PROBLEM 1 Mathew is baking a cake. His recipe calls for: $1\frac{1}{3}$ cups of flour, $1\frac{3}{4}$cups of milk, and $1\frac{1}{12}$ cups of sugar. He has two measuring cups that he inherited from uncle Archimedes. They measure 2/3 of a cup and 1/4 of cup. Can he bake his cake? If YES, what is the smallest number of measurements needed to make this cake? Please indicate in the work sheet the number of times Mathew used each measuring cup to obtain the required amount of flour, milk and sugar.

ANSWER: YES, Matthew can make this cake. He needs at least 9 measurements .

SOLUTION: We must use the measuring cups a total of nine times.

The $1\frac{1}{3}$ cups of flour may be measured by using the $\frac{2}{3}$ measuring cup twice, which we write as

$$2 \times \frac{2}{3}.$$

The $1\frac{3}{4}$ cups of milk may be expressed as

$$3 \times \frac{2}{3} - 1 \times \frac{1}{4},$$

where the minus means that one of the full $\frac{2}{3}$ cups was poured into the $\frac{1}{4}$ cup, leaving $\frac{2}{3} - \frac{1}{4} = \frac{5}{12}$ in the larger measuring cup, which was poured into the bowl. The "minus" cup is discarded.

Finally, we measure out the $1\frac{1}{12}$ cups of sugar as

$$2 \times \frac{2}{3} - 1 \times \frac{1}{4}.$$

This all requires that we use the measuring cups $2 + (3 + 1) + (2 + 1) = 9$ times, including the ones where we threw away ingredients.

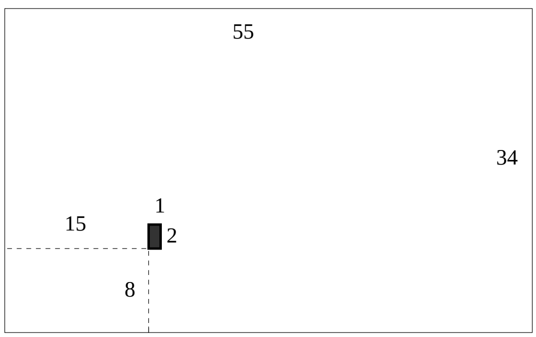
(These are the words of 9th grader Kristin Cordwell, Manzano High School, who kindly send me her solutions and those of her classmates to most of the problems.)

Addenda to the solution The above is the most "efficient" solution in terms of the least number of measurements required. However a natural choice for obtaining the $1\frac{3}{4}$ cups of flour is to use seven times the small measurement cup and express

$$1 + \frac{3}{4} = 7 \times \frac{1}{4}.$$

If we are only thinking additively then we will find it impossible to obtain the $1\frac{1}{12}$ cups of sugar and we might conclude, erroneously, that Matthew can not make this cake. At this point some might have realized that in order to get the right amount of sugar one has to use twice the largest cup and then use once the smaller cup to remove some sugar, but if this knowledge is not used to measure the flour, the conclusion would be that $2+7+2+1=12$ measurements are required to make the cake, but as shown before, twelve is not the least number of measurements required. Of course there are infinitely many ways of making the cake, if we put extra baking material in to be discarded later (for example: we add two cups of sugar using three times the large measuring cup, and we remove them using eight times the small measuring cup, and we keep on doing this as many times as we want).

PROBLEM 2 Builder Artemisa has to pave with tiles a rectangular patio with dimensions 34 units by 55 units. The patio has a rectangular fountain with dimensions 1 unit by 2 units that will not be paved. The exact location of the fountain is indicated in the picture below.

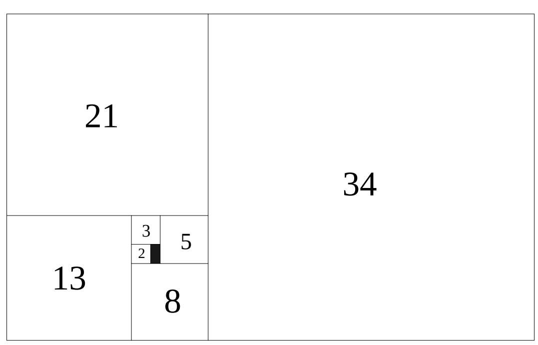


Artemisa's tile machine is pretty temperamental and *only produces square tiles of integer dimensions* (1 unit by 1 unit, 2 units by 2 units, etc.), moreover *once it produces a tile of a given size then it refuses to produce more of that size.* Fortunately Artemisa is very strong and she can handle square tiles of any dimension.

Will Builder Artemisa be able to cover her patio completely with square tiles of different sizes? (She is not allowed to break her tiles, pave beyond the patio, or to superimpose tiles). If the answer is YES, please show a paving in the map provided in the work sheet and write inside of each square tile its side length.

ANSWER: YES, Artemisa will be able to pave her patio.

SOLUTION: Here is a paving,



Some of you might have recognized the Fibonacci numbers, defined by the following recurrence rule: $F_0 = F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Using this rule we can compute the first Fibonnacci numbers:

$$\begin{aligned} F_2 &= F_1 + F_0 = 2, \\ F_3 &= F_2 + F_1 = 3, \\ F_4 &= F_3 + F_2 = 5, \\ F_5 &= F_4 + F_3 = 8, \\ F_6 &= F_5 + F_4 = 13, \\ F_7 &= F_6 + F_5 = 21, \\ F_8 &= F_7 + F_6 = 34, \\ F_9 &= F_8 + F_7 = 55. \end{aligned}$$

Part of the solution to Artemisa's paving problem is the fact that the sum of the areas of the square tiles used plus the area of the fountain must equal the area of the rectangle, that is

$$\underbrace{1^2 + 1^2}_{\text{the fountain}} + 2^2 + 3^2 + 5^2 + 8^2 + 13^2 + 21^2 + 34^2 = 34 \times 51.$$

In terms of the Fibonacci numbers, this can be written as,

$$F_0^2 + F_1^2 + F_2^2 + F_3^2 + F_4^2 + F_5^2 + F_6^2 + F_7^2 + F_8^2 = F_8 \times F_9.$$

One wonders whether this formula works in general or not. If we go back to the tiling we realize that this indeed works, furthermore, not only the sum of the squares of the first n Fibonacci numbers add up to the product of the n-th times the next Fibonacci number, that is, for all $n \geq 0$,

$$F_0^2 + F_1^2 + F_2^2 + \cdots + F_n^2 = F_n \times F_{n+1},$$

but also the corresponding square tiles can be arranged to cover completely a rectangle of area $F_n \times F_{n+1}$.

This was the basic property about Fibonacci numbers that was needed in order to solve Problem 8 in the Round II exam of last year's contest (Feb 2005). Two proofs were presented in the solutions to that exam (one in the spirit of Artemisa's tiling, another by induction). You can find them online

http://www.math.unm.edu/math_contest/sol04/solution_round2_04-05.pdf

PROBLEM 3 You have a dart board divided in two regions, one red, one black. If you hit the red region you get 7 points if you hit the black region you get 10 points. Can you get 83 points? Can you get 22 points? What is the largest number of points you can NOT get?

ANSWER: We can get 83 points. We can not get 22 points.
The largest number of points that we can NOT get is 53.

SOLUTION: We can get 83 points by hitting the black region twice, and hitting the red region nine times, that is $2 \times 10 + 9 \times 7 = 83$.

We cannot get 22 points. The closest combinations of 7 and 10 that are less than or equal to 22 are: $1\times 7+1\times 10=17$, $3\times 7=21$, and $2\times 10=20$. Each remainder is smaller than 7, so we cannot add 7 or 10 to make 22.

This is the "Chicken McNuggets" problem. If two number a and b are relatively prime, the largest number that cannot be made up of positive combinations of a and b is $ab-(a+b)$, the product minus the sum. For 7 and 10, this is $70-17=53$.

(These are the words of 9th grader Kristin Cordwell, Manzano High School.)

The "Chicken McNuggets" problem: If you were familiar with this problem you can indeed use its solution, but if you are not this remains pretty mysterious. Why "Chicken McNuggets"? Suppose at McDonnald's you can get boxes with a chicken McNuggets or b chicken McNuggets, and you want N chicken McNuggets, can you get them? In our example $a=7$, $b=10$, and if you wanted 83 chicken McNuggets you could get them but if you wanted 22 you could not. The question then is what is the largest number of chicken McNuggets that you cannot get? Clearly calling them points or chicken McNuggets makes no difference, the problem is the same.

Before we embark in a full blown proof of the general problem, let us solve our particular problem by hand and a little experimentation.

The smallest numbers we can achieve with positive combinations of 7 and 10 are:

$$7, 10, 14, 17, 20, 21, 24, 27, 28, 30, 31, 34, 35, 37, 38, 40, 41, 42, 44, 45, 47, 48,$$

$$49, 50, 51, 52, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, \ldots$$

It seems that every number afterwards can be achieved. Let us try to argue this.

Any number ending in 0 is a multiple of 10, say $10k$ and can be achieved by hitting the black region k times (or getting k boxes with 10 chicken McNuggets).

Any number ending on 7 is of the form $10k+7$ for some $k\geq 0$, and can be achieved by hitting the black region k times and the red region once.

The number 4 cannot be achieved, however, 14, 24, and any number of the form $10k+14=10(k+1)+4$ with $k\geq 0$ can be achieved by hitting the black region k times and the red region twice.

The numbers 1, 11 cannot be achieved, however, 21, 31, and any number of the form $10k+21=10(k+2)+1$ with $k\geq 0$ can be achieved by hitting the black region k times and the red region 3 times.

The numbers 8, 18 cannot be achieved, however, 28, 38, and any number of the form $10k+28=10(k+2)+8$ with $k\geq 0$ can be achieved by hitting the black region k times and the red region 4 times.

The numbers 5, 15, 25 cannot be achieved, however, 35, 45, and any number of the form $10k+35=10(k+3)+5$ with $k\geq 0$ can be achieved by hitting the black region k times and the red region 5 times.

The numbers 2, 12, 22, 32 cannot be achieved, however, 42, 52, and any number of the form $10k+42=10(k+4)+2$ with $k\geq 0$ can be achieved by hitting the black region k times and the red region 6 times.

The numbers 9, 19, 29, 39 cannot be achieved, however, 49, 59, and any number of the form $10k + 49 = 10(k + 4) + 9$ with $k \geq 0$ can be achieved by hitting the black region k times and the red region 7 times.

The numbers 6, 16, 26, 36, 46 cannot be achieved, however, 56, 66, and any number of the form $10k + 56 = 10(k + 5) + 6$ with $k \geq 0$ can be achieved by hitting the black region k times and the red region 8 times.

Finally, the numbers 3, 13, 23, 33, 43, and 53 cannot be achieved, however, 63, 73, and any number of the form $10k + 63 = 10(k + 6) + 3$ with $k \geq 0$ can be achieved by hitting the black region k times and the red region 9 times.

Any number $M > 60$ will be of the form $M = 10m + r$ where $r = 0, 1, 2, 3, 4, 5, 6, 7, 8$ or 9, and $m \geq 6$, and all these cases have been considered above and are achievable by positive combinations of 10 and 7. Therefore the largest number we cannot achieve is 53.

This is not the most short or elegant argument, but it gives us the answer for the particular case study. However given any pair of numbers a and b which do not have common divisors, we can do the same, can we? Well, the fact that in our case a was 10, facilitated the calculations. It is actually not that clear that we can repeat the argument.

Exercise: *Try to argue by looking at remainders like we just did in the general case.*

We will now verify that if a and b have maximum common divisor 1, then the largest number we can not reach by positive combinations of a and b is exactly $ab - a - b$.

We will assume known a very important fact in number theory:

The Euclidean Algorithm: *Given two positive integers a, b, let d be their maximum common divisor, then there exists integers p, q such that*

$$pa + qb = d.$$

Furthermore there is an algorithm (Euclid's Algorithm) that allows us to find p and q.

It should be clear that d is the smallest positive integer that can be found by adding integer multiples of a and b. Indeed, there exist positive integers m, n such that $a = dm$, $b = dn$ and their maximum common divisor is 1 (otherwise d will not be the m.c.d. of a and b), hence if $pa + qb = M$ then d must divide M this implies that $d \leq M$. This argument actually shows that it suffices to prove the Euclidean Algorithm in the case when a, b are relatively prime, that is when $d = 1$.

Exercise: *Prove the Euclidean algorithm.*

Proof of the Chicken McNuggets Problem: By the Euclidean Algorithm, there exist p, q integers such that

$$pa + qb = 1.$$

But of course one is positive and the other negative, that is $pq < 0$ (if $a = 1$ or $b = 1$ we are not in a very interesting case, we can reach every positive integer k just by multiplying a or b

by k, so we can assume $a, b > 1$.). For each M integer, we can find an integer solution $x = m$, $y = n$ to the equation,

$$ax + by = M, \tag{1}$$

namely set $x = Mp$ and $y = Mq$. But this will not be a positive combination of the numbers a and b, since $mn = M^2(pq) < 0$. Are there other integer solutions to (1)? Yes, there are. Equation (1) is the equation of a line cutting the X-axis at $x = M/a$ and the Y-axis at $y = M/b$. This line has negative slope equal to the rational number $-a/b$, and we have shown it passes through at least one point with integer coordinates, namely the point (Mp, Mq), but that point is in the second or fourth quadrant. Notice that if (x, y) is on the line (1), and we add b to the X-coordinate and subtract a from the Y-coordinate (or subtract b to the X-coordinate and add a from the Y-coordinate) we get another point on the line (1), in fact

$$a(x \pm b) + b(y \mp a) = ax \pm ab + by \mp ab = ax + by = M.$$

Since we have a point on the line with integer coordinates, by this procedure we find infinitely many other points on the line with integer coordinates, in fact any point of the form $x = Mp + kb$, $y = Mq - ka$ has integer coordinates and is on the line for k an integer, since

$$a(MP + kb) + b(Mq - ka) = aMP + abk + bMQ - abk = aMP + bMq = M.$$

Without loss of generality we can assume that the point (Mp, Mq) is in the fourth quadrant, that is $p > 0$, and $q < 0$, notice that this implies that $Mp > M/a$. In that case we want to move NW (that is subtract b from the x-coordinate and add a to the Y-coordinate). If we do this step by step, after a finite number of steps the X-coordinate will be smaller than M/a, hence the Y-coordinate will be positive. if we can guarantee that the X-coordinate is positive then we are in business. Let k be the smallest positive integer such that $Mp - kb < M/a$, this implies that $Mp - (k-1)b \geq M/a$, and as we just explained, necessarily, $Mq + ka > 0$. So far we have made no restriction on M. If $M > ab$ then clearly, $M/a > b$ this implies that $Mp - kb > 0$, otherwise, $Mp-(k-1)b = Mp-kb+b \leq b < M/a$ which is not possible. Also if $M = ab$ then the points $(b, 0)$ and $(0, a)$ are on the line and our problem is solved. So we have found an upper bound for the numbers M that we cannot achieve by positive combinations of a and b, namely $M < ab$.

At this point we prove that if $M > ab - a - b$ then we can find a positive combination of a and b that gives M, and if $M = ab - a - b$ we cannot. Therefore the largest number we cannot reach is indeed $ab - a - b$.

Suppose we can write $ab - a - b$ as a positive linear combination of a and b. Hence, there are integers $m, n \geq 0$ such that $ab - a - b = ma + nb$, and we conclude that

$$ab = (m + 1)a + (n + 1)b.$$

But since a and b are relatively prime, this implies that b divides $(m+1)$ and a divides $(n+1)$, hence $m + 1 \geq b$, and $n + 1 \geq a$, but these inequalities imply that

$$ab = (m + 1)a + (n + 1)b \geq ba + ab = 2ab,$$

and these is a contradiction since a, b are nonzero. The conclusion is that we cannot write $ab - a - b$ as a positive linear combination of a and b.

To show that numbers $M > ab - a - b$ can be written as positive linear combinations, it suffices to show the contrapositive, that is, if M cannot be written as a positive combination of a and b then necessarily $M \leq ab - a - b$. But this implies that we have two "consecutive" points with integer coefficients on the line such that one is in the second quadrant, and the other in the fourth quadrant. More precisely those solutions are exactly the points $(Mp - kb, Mq + ka)$, and $(Mp - (k-1)b, Mq + (k-1)a)$, where k is the smallest integer such that $Mp - kb < M/a \leq Mp - (k-1)b$. But since the first one is in the second quadrant it must be the case that $Mp - kb \leq -1$, and since the second is in the fourth quadrant it must be the case that $Mq + (k-1)a \leq -1$, that is $Mq + ka \leq a - 1$. Now we know both points are on the line (1), hence

$$M = (Mp - kb)a + (Mq + ka)b \leq (-1)a + (a-1)b = ab - a - b.$$

And we are done!

PROBLEM 4 (a) We are given 6 points on a circle equally spaced (think of a clock with just the even hours). How many triangles can be constructed so that their vertices are three of the given points on the circle? How many among those triangles are: isosceles triangles? equilateral triangles? right triangles?

(b) Same question as above, except that now you chose among 2005 equally spaced points on the circle.

ANSWER: (a) With 6 points we can construct 20 triangles.
There are 8 isosceles triangles, 2 equilateral triangles, 12 right triangles.
(b) With 2005 points we can construct 1,341,349,010 triangles.
There are 2,009,010 isosceles triangles, 0 equilateral triangles,
and 0 right triangles.

SOLUTION 1: (a) Since no three points are collinear, any choice of three will make the vertices of a triangle, so there are

$$\binom{6}{3} = \frac{6 \times 5 \times 4}{3 \times 2 \times 1} = 20, \qquad \text{triangles.}$$

For the isosceles triangles, we pick a starting vertex, then pick the other two points to be an equal distance away, going both directions. There are two such pairs of points , for each starting point, however, one of the two pairs will form an equilateral triangle, which means that if we just take $6 \times 2 = 12$, we will be over-counting. There are $6 \times 1 = 6$ non-equilateral isosceles triangles and two equilateral triangles, for a total of 8 isosceles triangles.

There are two equilateral triangles, corresponding to the two sets of three equidistant points.

Since a triangle that is circumscribed by a circle is a right triangle if and only if the hypotenuse is a diameter, we look at the possible diameters with endpoints on the six points.

There are three possible diameters, and each one has four possible points for the other vertex of the right triangle, so there are $3 \times 4 = 12$ right triangles.

(b) Similarly, since no three points are collinear, any choice of three will make the vertices of a triangle, so there are there are

$$\begin{aligned}\binom{2005}{3} &:= \frac{2005!}{(2005-3)!3!} = \frac{2005 \times 2004 \times 2003}{3 \times 2 \times 1} \\ &= 2005 \times 334 \times 2003 = 1,341,349,010, \quad \text{possible triangles.}\end{aligned}$$

For each starting point, we can pick the two points that are equidistant in each direction along the circle to form isosceles triangles. There are $\frac{2005-1}{2} = 1002$ such pairs of points, for each initial point. Thus, there are

$$2005 \times 1002 = 2,009,010, \quad \text{isosceles triangles.}$$

Since 2005 is not divisible by 3, we cannot span an angle of exactly 120°, so there are no equilateral triangles. This also shows that we did not overcount our isosceles triangles.

Since 2005 is not divisible by 2, there are no diameters of the circle between any two points. Since a right triangle requires that the hypotenuse be a diameter of the circumscribing circle, there are no right triangles.

(These are the words of 9th grader Kristin Cordwell, Manzano High School.)

SOLUTION 2: The above solution requires working knowledge of combinatorial numbers to count all possible triangles. Part (a) could have been done counting directly. Label the six points $P_1, P_2, P_3, P_4, P_5, P_6$ ordered clockwise, for example.

- Count how many triangles have P_1 as a vertex, to do that
  - count how many have P_2 also as a vertex, and there are exactly 4 options for the third vertex, the remaining four vertices P_3, P_4, P_5, P_6.
  - count how many have P_3 also as a vertex, but not P_2, and there are exactly 3 options for the third vertex, the remaining three vertices P_4, P_5, P_6.
  - count how many have P_4 also as a vertex, but not P_2 nor P_3, and there are exactly 2 options for the third vertex, the remaining two vertices P_5, P_6.
  - count how many have P_5 also as a vertex, but not P_2, P_3 nor P_4, and there is exactly 1 option for the third vertex, the remaining vertex P_6.

  Hence there are $4 + 3 + 2 + 1 = 10$ triangles that have P_1 as a vertex.

- Count how many triangles have P_2 as a vertex, but not P_1, to do that
  - count how many have P_3 also as a vertex (but not P_1), and there are exactly 3 options for the third vertex, the remaining three vertices P_4, P_5, P_6.

- count how many have P_4 also as a vertex, but not P_3 (nor P_1), and there are exactly 2 options for the third vertex, the remaining two vertices P_5, P_6.
- count how many have P_5 also as a vertex, but not P_3 nor P_4 (nor P_1) , and there is exactly 1 option for the third vertex, the remaining vertex P_6.

Hence there are $3 + 2 + 1 = 6$ triangles that have P_2 as a vertex but not P_1 (so we have not overcounted).

- Count how many triangles have P_3 as a vertex, but not P_2 or P_1, to do that
  - count how many have P_4 also as a vertex (but not P_1 or P_2), and there are exactly 2 options for the third vertex, the remaining two vertices P_5, P_6.
  - count how many have P_5 also as a vertex, but not P_4 (nor P_1 or P_2), and there is exactly 1 option for the third vertex, the remaining vertex P_6.

  Hence there are $2 + 1 = 3$ triangles that have P_3 as a vertex but not P_1 nor P_2 (so we have not overcounted).

- Finally there is just one triangle that has P_4 as a vertex but not P_1, P_2 ir P_3. The other two vertices have no other choice than to be P_5 and P_6.

Therefore there are a total of $10 + 6 + 3 + 1 = 20$ triangles.

Can we use this counting technique to count the number of triangles in part (b)? Yes, but it will be long and prone to mistakes, and it will require working knowledge of sums. If we could guess a closed formula for the number of triangles T_n given n points on the circle, then we could use this idea to verify the formula by mathematical induction, since it should be clear that the number of triangles given $n+1$ points is the number of triangles given n points plus the number of triangles that will have the $n+1$ point as a vertex, and those are exactly $1 + 2 + 3 + \cdots (n-2) = \frac{(n-2)(n-1)}{2}$ triangles, hence the following recurrence formula holds,

$$T_{n+1} = T_n + \frac{(n-2)(n-1)}{2}.$$

It can be verified that the combinatorial numbers

$$\binom{n}{3} := \frac{n!}{(n-3)!3!} = \frac{n(n-1)(n-2)}{6}$$

satisfy the recurrence relation, and that $\binom{3}{3} = 1 = T_3$ which is the number of different triangles you can construct if you were given 3 points on the circle, just one.

Kamil Adamczewski, 11th grader from United World College, realized that if n is odd, say $n = 2k + 1$ then

$$T_{2k+1} = 1 + 3^2 + 5^2 + \ldots + (2k-1)^2.$$

Exercise: *verify that the above claim is true and verify also that if* $n = 2k$ *then*

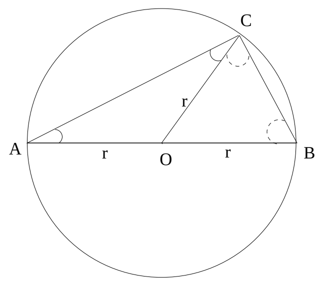
$$T_{2k} = 2^2 + 4^2 + \cdots + (2k-2)^2.$$

The following formula might be useful,

$$1^2 + 2^2 + 3^2 + \cdots + (n-1)^2 + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

To solve part (b) it was necessary to know the following geometric fact: *Right triangles with vertices on a circle must have a diameter for hypotenuse, conversely if a triangle with vertices on a circle has a side that coincides with a diameter of the circle, then it subtends a right angle.*

Exercise: *Verify this geometric fact. The following picture should help.*



PROBLEM 5

(a) Can you find a positive integer k so that the first two digits of 2^k are 65?

(b) Can you find a positive integer n so that the first digit of 2^n is 7?

ANSWER: **(a)** YES, $k = 16$ for example.
(b) YES, $n = 46, 56, 66, 76, 86, 96$ for example.

SOLUTION :

(a) $2^{16} = 64 \cdot 2^{10} = 65536$.
This is found by direct experimentation, 16 is the first power that has the required property.

(b) $2^{46} = 70,368,744,177,664$.

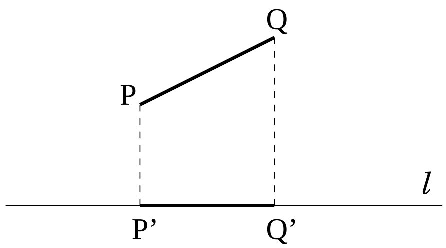
Note that $2^{10} = 1024$ will increase whatever it multiplies by 1000 plus 2.4%. Since 65, as in 65536, is close to a leading digit of 7, this might lead us to multiply by powers of $1024 = 2^{10}$. An estimate of the minimum necessary percentage increase from 6.5 to 7 is about 7.1% , and, since each 2^{10} will give 2.4% in addition to the factor of 1000, we would be led to try 2^{30} at a minimum. This gives us $2^{30} \cdot 2^{16} = 2^{46} = 70,368,744,177,664$.

It is easier, though, to also say that going from 6.5 to 8 (just bigger than 7) is something like 22%, which we might estimate to be about 8 or 9 times the 2.4%, if we include compounding. Thus, any power between 2^{46} and 2^{106} is going to have a leading digit of 7 or be close to it. We could be fairly certain, then, by picking numbers in the middle of 3 and 9, say, 5 or 6, that $2^{50} \cdot 2^{16} = 2^{66}$ and $2^{60} \cdot 2^{16} = 2^{76}$ will both have a leading seven, without needing to work out what the exact answer is.

(This solution was provided by 9th grader Chen Zhao, Manzano High School.)

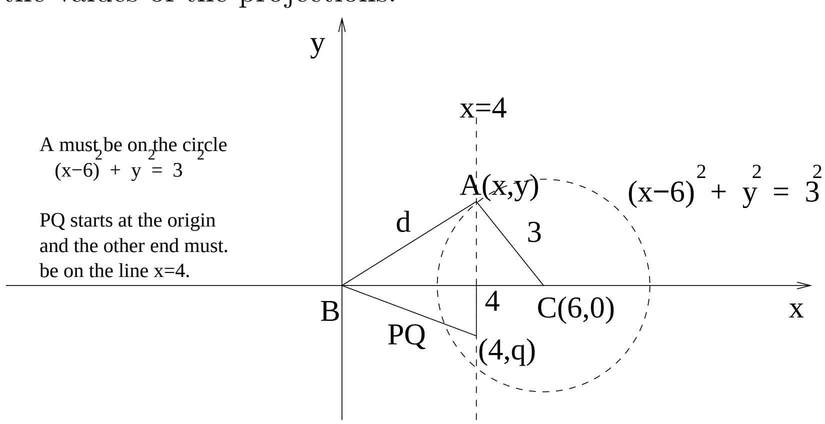
PROBLEM 6 We are given a line segment PQ and a triangle ABC in the same plane. Suppose the perpendicular projections of the segment PQ on the lines containing sides AB, BC, and CA, have lengths 3, 4, and 5, respectively. If the lengths of sides BC and CA are 6 and 3, respectively, find the length of the remaining side AB. Is the answer unique?

> **Note:** *The* perpendicular projection *of a line segment PQ on a line ℓ is the segment $P'Q'$ on the line ℓ found by dropping lines perpendicular to ℓ from P and Q, see figure below,*



ANSWER: THERE IS NO SOLUTION. The wording was very misleading. This problem was dropped from the exam, however it is a nice problem whose solution you will find below.

SOLUTION: Without loss of generality, we choose point B to be the origin, and point C to lie on the x-axis. We identify vector $\overrightarrow{A}$ with point A, and vector $\overrightarrow{C}$ with point C. $\overrightarrow{C}$ has coordinates (6,0), while we let the coordinates of $\overrightarrow{A} = (x, y)$. The distance of A from the origin we call d, which is what we want to find, and we know that vector $\overrightarrow{CA} = \overrightarrow{A} - \overrightarrow{C} = (x-6, y)$ has length 3, from the given information. We freely translate segment PQ so that one end is at the origin, and the other end has coordinates (p, q), where we may insist that $p \geq 0$ (otherwise we just slide the segment so that the other endpoint is at the origin, instead). This has no effect on the values of the projections.



The projection of $\overrightarrow{PQ}$ onto $\overrightarrow{C}$ (or onto the x-axis) has length 4, so we immediately have that $p = 4$, and $(p, q) = (4, q)$.

The projection of $\overrightarrow{PQ}$ onto $\overrightarrow{CA}$ may be expressed as (for those of you familiar with linear algebra and the dot product, see Problem 8, Solution 2)

$$\frac{\left|\overrightarrow{CA}\cdot\overrightarrow{PQ}\right|}{3} = \frac{|4(x-6)+qy|}{3} = 5.$$

This gives us

$$|4x+qy-24| = 15 \tag{2}$$

Similarly, the projection of $\overrightarrow{PQ}$ onto $\overrightarrow{A}$ may be expressed as

$$\frac{\left|\overrightarrow{A}\cdot\overrightarrow{PQ}\right|}{d} = \frac{|4x+qy|}{d} = 3.$$

This gives us

$$|4x+qy| = 3d \tag{3}$$

Equation (2) gives us two solutions

$$4x+qy = 39, \quad \text{and} \quad x+qy = 9.$$

Substituting these two values into equation (3) then gives us two possible solutions for d, namely

$$d = \frac{39}{3} = 13, \quad \text{and} \quad d = \frac{9}{3} = 3.$$

However, the first solution violates the triangle inequality, while the second solution gives a degenerate triangle $(3+3=6)$.

We conclude that there is no solution.

In the spirit of Ross Camp's PODASIP (Prove or disprove, and salvage if possible), we change the projected values to be $\overrightarrow{PQ}$ onto $\overrightarrow{C}$ to be 3, $\overrightarrow{PQ}$ onto $\overrightarrow{CA}$ to be 1, and $\overrightarrow{PQ}$ onto $\overrightarrow{A}$ to be 3, keeping the side lengths as originally specified. We then get $p=3$, so $\overrightarrow{PQ}=(3,q)$, and our equations become

$$\frac{\left|\overrightarrow{CA}\cdot\overrightarrow{PQ}\right|}{3} = \frac{|3(x-6)+qy|}{3} = 1,$$

which gives us

$$|3x+qy-18| = 3. \tag{4}$$

While the projection of $\overrightarrow{PQ}$ onto $\overrightarrow{A}$ may be expressed as

$$\frac{\left|\overrightarrow{A}\cdot\overrightarrow{PQ}\right|}{d} = \frac{|3x+qy|}{d} = 3.$$

This gives us

$$|3x+qy| = 3d \tag{5}$$

These two equations give us two solutions, $3d = 21$ and $3d = 15$, from which we get $d = 7$ or $d = 5$.

(This solution was provided by the New Mexico Math League.)

SOLUTION 2: Given segments ST and UV then the projection of ST over UV is the orthogonal projection of the segment ST on the line that contains the segment UV. We denote this projection $\text{Proj}_{UV}ST$, and its length is the length of ST times the cosine of the angle θ between the given segments, i.e.

$$|\text{Proj}_{UV}ST| = |ST| \times |\cos\theta|.$$

Similarly we can find the projection of UV over ST, denoted $\text{Proj}_{ST}UV$, and its length is the length of UV times the cosine of the angle θ between the given segments, i.e.

$$|\text{Proj}_{ST}UV| = |UV| \times |\cos\theta|.$$

Using this notation, and letting α, β and γ be the angles between PQ and BC, CA and AB respectively, the information we have is:

$$\begin{aligned}
|\text{Proj}_{AB}PQ| &= 3 = |PQ| \times |\cos\gamma|, \\
|\text{Proj}_{BC}PQ| &= 4 = |PQ| \times |\cos\alpha|, \\
|\text{Proj}_{CA}PQ| &= 5 = |PQ| \times |\cos\beta|.
\end{aligned}$$

These are three equations in four unknowns: $|PQ|$, $|\cos\gamma|$, $|\cos\alpha|$, and $|\cos\beta|$. Assuming that the length of PQ is neither zero nor infinity, we can solve for the absolute values of the cosines in terms of $|PQ|$, namely,

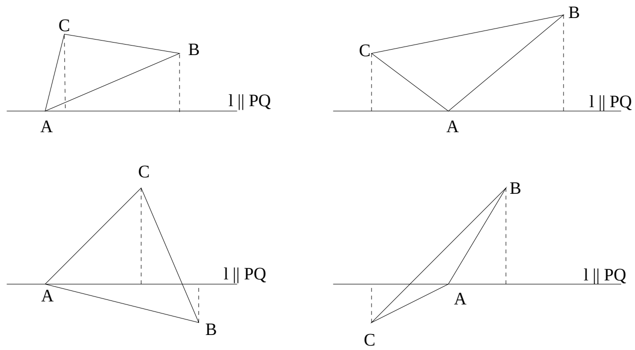
$$|\cos\gamma| = \frac{3}{|PQ|}, \quad |\cos\alpha| = \frac{4}{|PQ|}, \quad |\cos\beta| = \frac{5}{|PQ|}.$$

But the cosines also appear in the lengths of the projections of the sides of $\triangle ABC$ on segment PQ, and we know that $|BC| = 6$, and $|CA| = 3$, so

$$\begin{aligned}
\left|\text{Proj}_{PQ}AB\right| &= |AB| \times |\cos\gamma|, \\
\left|\text{Proj}_{PQ}BC\right| &= |BC| \times |\cos\alpha| = 6 \times |\cos\alpha|, \\
\left|\text{Proj}_{PQ}CA\right| &= |CA| \times |\cos\beta| = 3 \times |\cos\beta|.
\end{aligned}$$

Here the quantity we want to compute is $x = |AB|$. We seem to have introduced three new unknowns namely the lengths of the projections of the sides of the triangle onto PQ. However, because the segments we are projecting are sides of a triangle, it is not hard to see that the lengths of the projections must themselves satisfy some relation, in fact, no matter what configuration we have (below we picture a few possible configurations, where the line ℓ is parallel to the segment PQ), the length of the projection of one side is the sum or the difference of the other two, that is,

$$\left|\,\mathrm{Proj}_{PQ}BC\right| = \left|\,\left|\mathrm{Proj}_{PQ}AB\right| \pm \left|\mathrm{Proj}_{PQ}CA\right|\,\right|.$$



Therefore, given $\triangle ABC$ and a segment PQ, then the lengths of the projections of PQ over the sides of $\triangle ABC$, and the side lengths of $\triangle ABC$ must satisfy the following identity,

$$|\mathrm{Proj}_{AB}PQ| \times |AB| \pm |\mathrm{Proj}_{BC}PQ| \times |BC| \pm \left|\mathrm{Proj}_{PQ}CA\right| \times |CA| = 0, \tag{6}$$

where the sign depends on the particular configuration, and at least one sign is minus.

In our particular case, this equation reduces to,

$$(3 \times |AB|) \pm (4 \times 6) \pm (5 \times 3) = 0,$$

that is $3 \times |AB| = 39$ or 9, that is $AB = 13$ or $AB = 3$. But as explained the first is not a solution, the second is a degenerated solution. What is happening is that there is no such segment PQ that satisfies all the hypothesis.

If the values are changed, like in the modified problem in solution 1, then we will get that the possible solutions must satisfy,

$$(3 \times |AB|) \pm (3 \times 6) \pm (1 \times 3) = 0,$$

hence $|AB| = 5$ or $|AB| = 7$.

If you know about vectors, and you view each segment as a vector (directed segment, you put an arrow indicating orientation), then there is a natural orientation to be assigned to the projections, themselves can be viewed as vectors, and equation (6) about lengths becomes a statement about vectors, and the projection being a linear transformation. The sides of a triangle are connected by vector addition, the same holds for their projections onto PQ.

Addenda to the solution: Some comments are in order. Is it true that the solutions found using the linear algebra method described above that provide possible triangles are indeed solutions to the problem? What is clear is that if there is a solution to the problem then it must be also a solution of the linear algebra problem. And as indicated by Robert Cordwell, former winner of the Math contest, and now a freshman at CALTECH, solutions to the linear algebra problem that satisfy the triangle inequality do correspond to a physical solution, here is his argument.

For the particular example given at the end of Solution 1, with side lengths 6, 3, and 5, with point B at the origin and point C on the x-axis. We solved for d buy substituting the possible values of $3x + qy$ to get the length d which was our unknown. However to find a physical triangle, and the segment PQ, what we need are the coordinates of the point $A = (x, y)$, and the coordinate q to determine unequivocally the $\triangle ABC$, and the segment PQ.

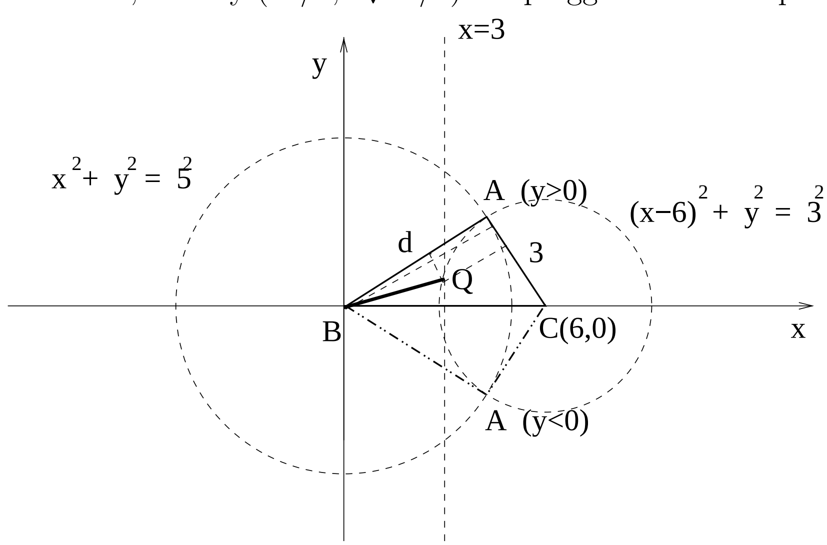
We can get a solution to the coordinates of point A by solving the simultaneous equations

$$\begin{aligned} x^2 + y^2 &= 25 \qquad \text{(or 49, for the other value of } d\text{)}, \\ (x-6)^2 + y^2 &= 9 \qquad \text{(point } A \text{ must lie on both these circles)}. \end{aligned}$$

The intersection of these two circles is two points (it must be two points, because the triangle inequality is satisfied; i.e., the sides are lengths of a possible non-degenerated triangle). Since $x^2 + y^2$ essentially cancel out, we easily solve for the x-coordinate of A, $x = 13/3$, and we get two possible solutions for y, $y = \pm 2\sqrt{14}/3$. We may pick the positive solution to y, as the other solution is just the reflection about the y-axis.

This can be done for any feasible triangle, so what is left to do is to check that the solution for the y-coordinate of Q that comes from equations (4) and (5) in Solution 1, gives a solution to the original problem. Recall that one endpoint of PQ, say P (if necessary after interchanging the roles of P and Q), lies on the origin, and that the x-coordinate of Q is fixed by the value of the projection of PQ onto the x-axis (side BC of the triangle).

Now, by the choice (solution) of d (we are working with $d = 5$), if equation (5) is satisfied, so will be equation (4), with $3x + qy$ being positive for equation (4) to work. Therefore, we can solve for q by setting $3x + qy = 15$. This gives $q = 3/\sqrt{14}$. If the value $Q = (3, 3/\sqrt{14})$ and the coordinates for A, namely $(13/3, 2\sqrt{14}/3)$ are plugged into the equations, everything works out.



Exercise: *Similarly when $d = 7$, find the possible values of x, y, and q in this case.*

So... does this carry over to a general argument? We think so!... The projection onto BC

is automatic, by the choice of p. Equations (4) and (5) basically are the requirements that the projections onto the other two sides be correct. The solutions for d, the length of the third side, tell us whether or not viable triangles can be formed. If they can, then we can calculate (x, y) (point A) by intersecting the two circles of radius equal to the other two side lengths. This will always have two solutions, if the triangle inequalities are satisfied. Plugging in the values of x and y to equations (4) (with 3 replaced by p) and (5) will give an equation for q, which can (always) be solved. Equation (4) will determine whether or not $px + qy$ will be positive or negative. Solving equation (5) for q will automatically give a proper solution for (4). For each possible value of x, y there is a solution for q, hence a segment PQ, and there are at most 2 possible solutions. It could happen that there is only one viable solution for d, and corresponding segment PQ with the required projection values.

SOLUTION 3 (Prof. L.-S. Hahn): The problem suggests there must be a relation between the lengths of projections of a segment and the side lengths of the triangle. Indeed the following theorem holds. More precisely, equation (6) must hold, but the proof we present here is vey elegant and purely geometric, no linear algebra involved!!!

Theorem: *Suppose PQ is a line segment on the plane of $\triangle ABC$. Let the perpendicular projections of point P on (the extensions of) sides BC, CA, AB be D, E, F, and those of point Q be L, M, N. Then*

$$\overline{BC} \times \overline{DL} + \overline{CA} \times \overline{EM} + \overline{AB} \times \overline{FN} = 0,$$

where $\overline{BC} \times \overline{DL} > 0$ if $\overrightarrow{BC}$ and $\overrightarrow{DL}$ have the same direction, but $\overline{BC} \times \overline{DL} < 0$ if they have opposite direction, and $\overline{BC} \times \overline{DL} = 0$ if the points D and L coincide. Similarly for the other products, $\overline{CA} \times \overline{EM}$, $\overline{AB} \times \overline{FN}$. [1]

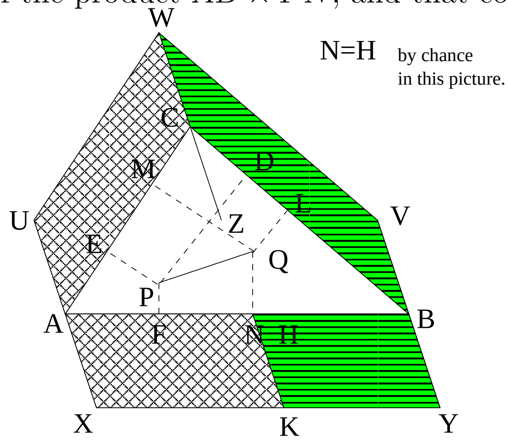
Proof: Draw line segments UX, VY, and WZ that are perpendicular to PQ having the lengths twice that of PQ with A, B, C as their respective midpoints. Discard the portion that is inside $\triangle ABC$ (see picture at the end of the proof, in the picture the segment we are discarding is segment CZ).

Note that if PQ is not perpendicular to any of the three sides of $\triangle ABC$, then there is bound to be one segment half of which is inside the triangle. This is easy to verify because the sum of the three angles of a triangle is π, and so if we parallel translate three angles so that their vertices meet at one point, then the union of the three covers half of the plane. Therefore, regardless of the direction of the line segment whose midpoint is at the common vertex of the angles, half of the segment must be in the half of the plane covered by the union of these three angles. (And the area of the parallelogram on the opposite side will be the sum of that of the other two.) The case that PQ is perpendicular to one of the three sides of a triangle is left to the reader.

Now it is easy to see that the height of parallelogram $BVWC$ on side BC is equal to the length of the projection DL of PQ on BC. Hence the absolute value of the product

[1] Note that at least one of the products is negative and one is positive. We could have rewritten this as equation (6), with ordinary length, here the notation $\overline{AB}$ denotes a "signed" length, where the sign is determined as described in the theorem.

$\overline{BC} \times \overline{DL}$ is equal to the area of the parallelogram $BVWC$, which, in turn, is equal to that of parallelogram $BHKY$, where H, K are the intersection of the extensions of WC with AB and XY, respectively. Similarly, the absolute value of product $\overline{CA} \times \overline{EM}$ is equal to the area of the parallelogram $CWUA$, which in turn is equal to that of parallelogram $AXKH$. But the parallelograms $BHKY$ and $AXKH$ tile the parallelogram $ABXY$ whose total area is equal to the absolute value of the product $\overline{AB} \times \overline{FN}$, and that concludes the proof.



Exercise: *Can you produce a geometric argument that will ensure that if you have side lengths satisfying the triangle inequality, and projection lengths that satisfy the theorem, then there is a triangle, and there is a segment PQ that have the given properties. In fact, how many solutions are there (non-congruent).*

PROBLEM 7 What is the remainder when

$$P(x) = 1 - x + 2x^4 - 3x^9 + 4x^{16} - 5x^{25} + 6x^{36} + x^{2005}$$

is divided by $D(x) = x^2 - 1$?

> **Note:** *The* remainder *is a polynomial $R(x)$ of degree smaller than the* divisor $D(x)$ *such that there is another polynomial $Q(x)$, the* quotient, *such that*
>
> $$P(x) = Q(x)D(x) + R(x).$$
>
> *Given a polynomial $P(x)$ and a divisor $D(x)$ then the remainder and the quotient are uniquely determined. For example, if $P(x) = x^3 + 2x^2 - 4$, and we divide by $D(x) = x^2 - 1$, then the quotient and remainder are $Q(x) = x + 2$, $R(x) = x - 2$, since*
>
> $$x^3 + 2x^2 - 4 = (x + 2)(x^2 - 1) + (x - 2).$$

ANSWER: The remainder is $13 - 8x$.

SOLUTION 1: Since $D(x) = x^2 - 1$ is degree 2, the remainder polynomial must be of degree 1 or lower, so we may write it as $R(x) = a \cdot x + b$.

Note that $D(x)$ has roots ± 1.
Since $P(x) = Q(x) \times D(x) + R(x) = Q(x)D(x) + ax + b$, we know that

$$P(1) = Q(1)D(1) + a + b = Q(1) \times 0 + a + b = a + b,$$

because 1 is a root of $D(x)$. Similarly,

$$P(-1) = Q(-1)D(-1) - a + b = 0 - a + b = -a + b.$$

Computing $P(1) = 5$ and $P(-1) = 21$ directly from the form of $P(x)$, we have

$$\begin{aligned} a + b &= 5 \\ -a + b &= 21 \end{aligned}$$

This is easily solved for $b = 13$ and $a = -8$. We have $R(x) = -8x + 13$.

(This solution was provided 9th grader Alex Wiele, Manzano High School.)

SOLUTION 2: Notice that

$$\begin{aligned} y^2 - 1 &= (y-1)(y+1), \\ y^3 - 1 &= (y-1)(y^2 + y + 1), \\ \vdots &= \vdots \\ y^n - 1 &= (y-1)(y^{n-1} + y^{n-2} + \cdots + y^2 + y + 1. \end{aligned}$$

Therefore if we set $y = x^2$ then we get that,

$$x^{2n} - 1 = (x^2 - 1)\left(x^{2(n-1)} + x^{2(n-2)} + \cdots + x^4 + x^2 + 1\right),$$

that is the remainder when dividing an even monomial x^{2n} by $(x^2 - 1)$ is always 1,

$$x^{2n} = (x^2 - 1)Q_n(x) + 1, \quad Q_n(x) = x^{2(n-1)} + x^{2(n-2)} + \cdots + x^2 + 1.$$

As for the odd powers, notice that $x^{2n+1} - x = x(x^{2n} - 1)$, hence

$$x^{2n+1} = x(x^{2n} - 1) + x = x(x^2 - 1)Q_n(x) + x,$$

so the remainder when dividing any odd monomial by $(x^2 - 1)$ is x.

The remainder of any polynomial when dividing by $(x^2 - 1)$ is of the form $ax + b$. To find the remainder all we have to do is add the coefficients of the odd powers to obtain a and add the coefficients of the even powers to obtain b. In our case,

$$a = -1 - 3 - 5 + 1 = -8, \qquad b = 1 + 2 + 4 + 6 = 13,$$

therefore the remainder is $13 - 8x$.

Note: There is an algorithm to divide polynomials very similar to the division algorithm we use to divide integers. However, due to the large degree of the polynomial (2005), it will not be practical to use it in our case.

PROBLEM 8 We are given rectangles A and B with corresponding side lengths a_1, a_2 and b_1, b_2. Let C be the rectangle with side lengths a_1, b_2, denote its diagonal length c. Let D be the rectangle with side lengths a_2, b_1 denote its diagonal length d. Let R be the rectangle with side lengths c, d.

(a) Can the sum of the areas of A and B equal the area of R?

(b) Can the sum of the areas of A and B be strictly larger than the area of R?

(c) Can the sum of the areas of A and B be strictly smaller than the area of R?

If you answered YES to any of the above questions, please write down an example in the work sheet.

ANSWER: (a) YES, it can be equal. In fact any example where $a_1b_1 = a_2b_2$ will work.
(b) NO, it cannot be strictly larger.
(c) YES, it can be smaller. In fact any example where $a_1b_1 \neq a_2b_2$ will work.

SOLUTION: We set the area of A equal to $a_1 \times a_2$, and that of B equal to $b_1 \times b_2$. The length $c = \sqrt{a_1^2 + b_2^2}$, and $d = \sqrt{a_2^2 + b_1^2}$. The area of R is then equal to $c \times d$. We write the unknown relationship between the sum of the areas of A and B and the area of R as "twiddle", $\sim$, where we will determine whether $\sim$ is $=$, $>$, or $<$. So we write

$$\text{area}(A) + \text{area}(B) \sim \text{area}(R).$$

Since everything is positive, we preserve the relationship if we square both sides, which gives us

$$a_1^2a_2^2 + b_1^2b_2^2 + 2a_1a_2b_1b_2 \sim c^2d^2 = a_1^2a_2^2 + b_1^2b_2^2 + a_1^2b_1^2 + a_2^2b_2^2.$$

Subtracting the first two terms from both sides yields

$$2a_1a_2b_1b_2 = 2(a_1b_1)(a_2b_2) \sim (a_1b_1)^2 + (a_2b_2)^2.$$

Substituting $x = a_1b_1$ and $y = a_2b_2$, we have

$$2xy \sim x^2 + y^2, \quad \text{or} \quad 0 \sim x^2 + y^2 - 2xy = (x-y)^2.$$

Since all steps are reversible, we have

$$\text{area}(A) + \text{area}(B) \sim \text{area}(R) \quad \text{if and only if} \quad 0 \sim (x-y)^2.$$

Now it is fairly easy to answer the questions.

(a) The areas are equal only if $0 = (x-y)^2$, or only if $x = y$. This translates to $a_1b_1 = a_2b_2$, or when $\frac{a_1}{a_2} = \frac{b_2}{b_1}$, or when the two original rectangles are similar, but the second width is proportional to the first length and the second length is proportional to the first width. For example, if rectangle A is 1×2 and B is 6×3, then C is 1×3 with diagonal length $c = \sqrt{10}$, D is 2×6, with diagonal length $d = 2\sqrt{10}$. The area of R is then $2 \cdot 10 = 20$, which equals the sum of the areas of A and B.

(b) Since $0 \leq (x-y)^2$, we cannot have the sum of the areas of A and B ever being greater than the area of R.

(c) Yes, most of the time. All that we need to do is pick $x \neq y$, such as A is 3×1 and B is 1×4. Then the area of R is $5\sqrt{2}$, but the areas of A and B add up to 7 (which is less).

(This solution was provided by 9th grader Kristin Cordwell, Manzano High School).

SOLUTION 2: We are trying to compare $a_1a_2 + b_1b_2$ with $\sqrt{a_1^2 + b_2^2} \times \sqrt{a_2^2 + b_1^2}$.

Consider the points which have cartesian coordinates $P = (a_1, b_2)$, $Q = (a_2, b_1)$. Let 0 be the origin. Notice that the distance from P to O, is the length $\sqrt{a_1^2 + b_2^2} = |P|$, and the distance from Q to O is the length $\sqrt{a_2^2 + b_1^2} = |Q|$. Define the *dot product* between P and Q to be

$$P \cdot Q = a_1a_2 + b_1b_2.$$

In this notation, what we are trying to compare are $P \cdot Q$ and $|P| \times |Q|$. Those of you familiar with linear algebra might at this point know the answer, for all pairs of points P and Q in the plane, regardless of the sign of their coordinates,

$$|P \cdot Q| \leq |P| \times |Q|,$$

which is the simplest incarnation of a very famous inequality in mathematics, the *Cauchy-Schwarz Inequality.*

To prove it, we consider the angle θ between the segments OP and OQ. If we can show that

$$\cos\theta = \frac{P \cdot Q}{|P| \times |Q|}. \tag{7}$$

then we are done, because $|\cos\theta| \leq 1$. Moreover we get equality if and only if $\cos\theta = \pm 1$ which happens if and only if $\theta = 0$ or $\theta = \pi$. In our case, since a_1, a_2, b_1, b_2 are all positive numbers, the only possible scenario is $\theta = 0$, in that case we conclude that

$$|P| \times |Q| = P \cdot Q$$

and working out the algebra like in the first proof shows that this happens if and only if $a_1b_1 = a_2b_2$.

To prove equation (7) we refer to the following diagram. We will prove it in the case where P and Q are in the first quadrant (that is a_1, a_2, b_1, b_2 are all positive numbers).

Let θ_1 be the angle between the segment OP and the X-axis, let θ_2 be the angle between the segment OQ and the X-axis. Assume $\theta_1 < \theta_2$ (otherwise interchange the labels of the points) Then $\theta = \theta_2 - \theta_1$, and we can compute the cosine of the difference of two angles, in terms of the sines and cosines of te two angles, more precisely,

$$\cos\theta = \cos(\theta_2 - \theta_1) = \cos\theta_2 \cos\theta_1 + \sin\theta_2 \sin\theta_1.$$

We can compute in terms of the coordinates of P and Q the trigonometric functions on the right hand side of the above equation,

$$\cos\theta_2 = \frac{a_2}{|Q|}, \quad \sin\theta_2 = \frac{b_1}{|Q|}, \quad \cos\theta_1 = \frac{a_1}{|P|}, \quad \sin\theta_1 = \frac{b_2}{|P|}.$$

Therefore,

$$\cos\theta = \frac{a_2}{|Q|} \times \frac{a_1}{|P|} + \frac{b_1}{|Q|} \times \frac{b_2}{|P|} = \frac{a_2 a_1 + b_2 b_1}{|P| \times |Q|} = \frac{|P| \cdot |Q|}{|P| \times |Q|}.$$

Exercise: *Verify it is true also when the points P and Q are not both in the first quadrant.*

***

Dear students: If you have any suggestions about the Contest, or if you have different solutions to any of this year's problems, please send them to:

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or e-mail them to: *crisp@math.unm.edu*

Remember that you can find information about past contests at:
http://www.math.unm.edu/math_contest/contest.html

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