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THE KÄHLER GEOMETRY OF BOTT MANIFOLDS

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Abstract. We study the Kähler geometry of stage n Bott manifolds, which can be viewed as n-dimensional generalizations of Hirzebruch surfaces. We show, using a simple induction argument and the generalized Calabi construction from [ACGT04, ACGT11], that any stage n Bott manifold M_n admits an extremal Kähler metric. We also give necessary conditions for M_n to admit a constant scalar curvature Kähler metric. We obtain more precise results for stage 3 Bott manifolds, including in particular some interesting relations with c-projective geometry and some explicit examples of almost Kähler structures.

To place these results in context, we review and develop the topology, complex geometry and symplectic geometry of Bott manifolds. In particular, we study the Kähler cone, the automorphism group and the Fano condition. We also relate the number of conjugacy classes of maximal tori in the symplectomorphism group to the number of biholomorphism classes compatible with the symplectic structure.

Contents

1. Bott manifolds 5
1.1. Bott towers and their cohomology rings 5
1.2. The quotient construction and examples 6
1.3. Toric description of Bott manifolds 8
1.4. Equivalences and the Bott tower groupoid 10
2. The topology of Bott manifolds 13
2.1. Topological twist 13
2.2. Cohomological rigidity of Bott manifolds 14
2.3. Topological classification of stage 3 Bott manifolds 17
3. The complex viewpoint 19
3.1. The automorphism group 19
3.2. Divisors, support functions and primitive collections 21
3.3. The ample cone and Kähler cone 22
3.4. Fano Bott manifolds 26
4. The symplectic viewpoint 29
4.1. Toric symplectic manifolds and finiteness 29
4.2. Symplectic products of spheres 30
4.3. Further symplectic equivalences 34
5. The generalized Calabi and admissible constructions 35
5.1. The generalized Calabi construction 35
5.2. The admissible construction 37

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5.3. Applications of the admissible construction to Bott manifolds 38
5.4. Relation with c-projective geometry 40
5.5. Extremal almost Kähler metrics on certain stage 3 Bott manifolds 45
References 48

Introduction

The purpose of this paper is to present and develop the Kähler geometry of a class of toric complex manifolds known as *Bott manifolds*, with one ultimate goal being to understand their extremal Kähler metrics.

The introduction to Grossberg's PhD thesis [Gro91] describes a conjecture by Bott, in a 1989 letter to Atiyah, that the well-studied Bott–Samelson manifolds [BS58] "should be realizible as some kind of tower of projectivized vector bundles". The thesis then proved the conjecture, showing that each bundle in the tower is a $\mathbb{CP}^1$-bundle with a circle action, and Grossberg named such iterated $\mathbb{CP}^1$-bundles *Bott towers*. Their study was taken up by Grossberg and Karshon [GK94], who proved that isomorphism classes of Bott towers of complex dimension n are in bijection with $\mathbb{Z}^{n(n-1)/2}$. More precisely, given an integer-valued $n \times n$ lower triangular unipotent matrix $A = (A_i^j)$, they constructed a Bott tower $M_n(A)$ as quotient of $(\mathbb{C}_*^2)^n$ (with $\mathbb{C}_*^2 = \mathbb{C}^2 \setminus \{0\}$) by the action of a complex n-torus determined by A. Then they proved that there is a unique such Bott tower $M_n(A)$ in each isomorphism class.

A (stage n) Bott manifold M_n is a complex n-manifold biholomorphic to the total space of a Bott tower. Bott manifolds are natural generalizations of Hirzebruch surfaces: as shown by Masuda and Panov [MP08] they are precisely the toric complex manifolds whose fan Σ is the cone over an n-cross (i.e., combinatorially dual to an n-cube). Thus the primitive generators of the rays of Σ in the toric real Lie algebra τ come in opposite pairs $u_j, v_j : j \in \{1, \dots n\}$, and a subset of generators spans a cone of Σ iff it contains no opposite pairs.

The close relation to Hirzebruch surfaces suggests that one may be able to systematically explore the existence of extremal Kähler metrics, elegantly constructed by Calabi [Cal82] when $n = 2$, for arbitrary Bott manifolds.

Generally, let $\mathcal{K}(M_n)$ be the Kähler cone of a complex manifold M_n, and let $\mathcal{E}(M_n)$ be the subset of Kähler classes which contain an extremal Kähler metric.

Problem. *Describe the extremal Kähler cone $\mathcal{E}(M_n)$. In particular,* (1) *is $\mathcal{E}(M_n)$ nonempty, and if so,* (2) *is $\mathcal{E}(M_n) = \mathcal{K}(M_n)$?*

By a well-known result of LeBrun and Simanca [LS93], $\mathcal{E}(M_n)$ is open in $\mathcal{K}(M_n)$, a result which fails for the subclass of constant scalar curvature (CSC) Kähler metrics.

Question (2) is known in the affirmative for Hirzebruch surfaces [Cal82] (as already noted), and question (1) is true for toric surfaces in general [WZ11]. In another direction, Zhou and Zhu [ZZ08] proved that the existence of an extremal Kähler metric in the class $c_1(L)$ implies the K-polystability of the polarized toric complex manifold (M_n, L), and this has been generalized to arbitrary polarized complex manifolds by Stoppa and Székelyhidi [SS11] and Mabuchi [Mab14]. Now Donaldson [Don02] and Wang and Zhou [WZ11] have given examples of K-unstable Kähler classes on certain

smooth toric surfaces, so they admit no extremal metrics, and hence question (2) is false for toric surfaces in general. We refer to [WZ14] for a recent survey.

For Bott manifolds, our main general result is the following.

Theorem 1. *Let $M_n(A)$ be the Bott tower corresponding to the matrix A.*

(1) *An invariant $\mathbb{R}$-divisor D is ample if and only if its support function ψ_D satisfies*

$$\psi_D(u_j) + \psi_D(v_j) > -\psi_D\Big(-\sum_{i=j+1}^{n} A_i^j v_i\Big)$$

for all opposite pairs v_j, u_j of generators of the fan. In particular, if $A_i^j \leq 0$ for all $i > j \geq 1$ then the ample cone $\mathcal{A}(M_n)$, and hence the Kähler cone $\mathcal{K}(M_n)$, is the entire first orthant with respect to the natural basis of toric semi-ample divisors.

(2) *The extremal Kähler cone $\mathcal{E}(M_n(A))$ is a nonempty open cone in $\mathcal{K}(M_n(A))$, i.e., $M_n(A)$ admits extremal Kähler metrics.*

(3) *If the elements below the diagonal in any one row of the matrix A all have the same sign and are not all zero (in particular, if $A_2^1 \neq 0$), then $M_n(A)$ does not admit a CSC Kähler metric.*

The first part of this result follows from Batyrev's theorem for toric varieties [Bat91, CvR09] and the structure of the fan of $M_n(A)$. This result has been well known since Calabi [Cal82] for $\mathbb{CP}^1$ and Hirzebruch surfaces. In particular, with respect to a natural basis, the Kähler cone of any Hirzebruch surface is the entire first quadrant; however, this is not true generally for every Bott manifold. In fact, it breaks down at stage 3 as we shall show. We also mention recent related papers [Cha17b, Cha17a] which study the Mori cone, and toric degenerations, respectively.

For the second part, we apply a theorem of [ACGT11] obtained using the generalized Calabi construction of [ACGT04].

The third part uses the generalization of Matsushima's theorem by Lichnerowicz [Lic58] that the existence of a constant scalar curvature Kähler metric implies that $\mathrm{Aut}(M_n(A))$ is reductive. The criterion then follows from a balancing condition due to Demazure [Dem70] on the roots of the Lie algebra $\mathfrak{aut}(M_n(A))$ of $\mathrm{Aut}(M_n(A))$.

Theorem 1 does not produce explicit examples of extremal Kähler metrics on Bott manifolds; however, there are a few explicit examples known. These include:

(1) The product metrics on $\mathbb{CP}^1 \times \cdots \times \mathbb{CP}^1$;
(2) Calabi's extremal Kähler metrics [Cal82] on Hirzebruch surfaces (stage 2 Bott manifolds);
(3) Koiso and Sakane's Kähler–Einstein metric [KS86] on $\mathbb{P}(\mathbb{1} \oplus \mathcal{O}(-1,1)) \to \mathbb{CP}^1 \times \mathbb{CP}^1$; and, more generally,
(4) admissible extremal Kähler metrics on $\mathbb{P}(\mathbb{1} \oplus \mathcal{O}(k_1, \ldots, k_{n-1})) \to (\mathbb{CP}^1)^{n-1}$ whose existence was proven by Hwang [Hwa94] and Guan [Gua95] (see also [ACGT08a, Section 3] for a treatment using the admissible convention).

It is shown in [Hwa94] that for $n = 3$, if k_1, k_2 in (4) above have opposite signs then the corresponding twist ≤ 1 Bott manifold M_3 admits a CSC Kähler metric; whereas, if they have the same sign M_3 cannot have a CSC Kähler metric [ACGT08a].

To organize and generalize such examples, we note that the number $t \in \{0, \dots, n-1\}$ of holomorphically nontrivial $\mathbb{CP}^1$ bundles in a Bott tower $M_n(A)$ is a biholomorphism invariant of the total space which we call the (holomorphic) *twist*. An analogous topological invariant (the number of topologically nontrivial $\mathbb{CP}^1$ bundles in the tower), which we call the *topological twist*, was introduced by Choi and Suh in [CS11]. As the above examples are all $\mathbb{CP}^1$ bundles over a product $(\mathbb{CP}^1)^{n-1}$, they all have twist ≤ 1.

There is a dual notion of (holomorphic or topological) *cotwist* $t' \in \{0, \dots, n-1\}$, which is the number of bundles in the tower such that the inverse image of any $\mathbb{CP}^1$ fiber in $M_n(A)$ is (holomorphically or topologically) trivial over the fiber. For example, a stage 3 Bott manifold M_3 may be considered as a bundle of Hirzebruch surfaces over $\mathbb{CP}^1$, or as a $\mathbb{CP}^1$ bundle over a Hirzebruch surface. If M_3 has twist 1, it is a $\mathbb{CP}^1$ bundle over $\mathbb{CP}^1 \times \mathbb{CP}^1$, while if it has cotwist 1, it is a $\mathbb{CP}^1 \times \mathbb{CP}^1$ bundle over $\mathbb{CP}^1$.

In Section 1, after reviewing Bott towers and the quotient construction, we present the Bott manifolds of twist $t = 0, 1, 2$, which we use as running examples throughout the article. In particular, these values of t cover all stage 3 Bott manifolds, on which we place much emphasis, $n = 3$ being the next dimension after Hirzebruch surfaces. We also discuss when two stage n Bott manifolds $M_n(A)$ and $M_n(A')$ are biholomoprhic.

In addition to proving Theorem 1 and giving examples, we place them in context in order to motivate further study. Bott towers can be studied as smooth manifolds, complex manifolds, symplectic manifolds or Kähler manifolds, and the rest of the article explores these approaches. Along the journey, we obtain several results of independent interest, as we now explain.

In Section 2 we consider the topology of Bott manifolds. In recent years, research on Bott manifolds has centered around the *cohomological rigidity problem* [MP08, CMS11, CM12, Cho15] which asks if the integral cohomology ring of a toric complex manifold determines its diffeomorphism (or homeomorphism) type. This problem is still open even for Bott manifolds, but has an affirmative answer in important special cases: in particular for $n \leq 4$ and for topological twist $t \leq 1$, the cohomology ring determines the diffeomorphism type [CMS10, CS11, CM12, Cho15]. Hence, in these cases the homeomorphism classification coincides with the diffeomorphism classification. We review these ideas and obtain a diffeomorphism classification of stage 3 Bott manifolds.

Theorem 2. *The diffeomorphism type of a stage* 3 *Bott manifold M_3 is determined by its second Stiefel–Whitney class $w_2(M_3)$ and its first Pontrjagin class $p_1(M_3)$, which can be any integral multiple p of the primitive class $\frac{1}{2}c_1(\mathcal{O}(1,1))^2$. Moreover*:

(1) *Each diffeomorphism type contains finitely many Bott manifolds with twist ≤ 1, determined by the prime decomposition of p.*
(2) *There are precisely three diffeomorphism types of stage* 3 *Bott manifolds M_3 of cotwist ≤ 1 and each diffeomorphism type has an infinite number of inequivalent toric Bott manifolds.*

In Section 3, we study the automorphism group, the Kähler cone, and the Fano condition. For stage 3 Bott manifolds we can give specific information about $\mathcal{E}(M_n)$ and CSC Kähler metrics according to the diffeomorphism type. For example, using the results in [Hwa94, Gua95] we obtain:

Theorem 3. *A stage* 3 *Bott tower* $M_3(A)$ *admits a constant scalar curvature Kähler metric if and only if* $\mathrm{Aut}(M_3(A))$ *is reductive, which holds if and only if* $A_2^1 = 0$ *and* $A_3^1 A_3^2 < 0$ *or* $A = I$.

We turn to the symplectic viewpoint in Section 4. In Proposition 4.2, we observe that a result of McDuff [McD11] implies that there are only finitely many biholomorphism classes of complex structure compatible with a fixed compact toric symplectic manifold.

Theorem 4. *Let* (M, ω) *be a symplectic* $2n$*-manifold. Then there are finitely many biholomorphism classes of Bott manifolds compatible with* (M, ω)*, and their number is bounded above by the number of conjugacy classes of* n*-tori in the symplectomorphism group* $\mathrm{Symp}(M, \omega)$.

In Section 5, we use the generalized Calabi construction to prove Theorem 1 (2), and study the admissible construction. Generally, Bott manifolds do not fit so well with the admissible construction (cf. [ACGT08a] and references therein) that has been so successful in producing explicit examples of extremal Kähler metrics. Indeed only the examples of twist ≤ 1 use the admissible construction. Nevertheless, in Section 5.3 we describe such examples in arbitrary dimension, and these suffice to show that for stage 3 Bott manifolds of twist ≤ 1 there is an extremal Kähler metric in every Kähler class, i.e., $\mathcal{E}(M_3) = \mathcal{K}(M_3)$ in this case.

In Section 5.4 we touch briefly on how admissible examples are related to c-projective equivalence [CEMN16]. In particular we prove:

Theorem 5. *On the total space of* $\mathbb{P}(\mathbb{1} \oplus \mathcal{O}(-1,1)) \to \mathbb{CP}^1 \times \mathbb{CP}^1$ *there exists an infinite number of pairs of c-projectively equivalent constant scalar curvature (CSC) Kähler metrics which are* not *affinely equivalent.*

We end with some almost Kähler examples, in the spirit of [Don02, ACGT11, Lej10].

1. Bott manifolds

We begin by following Grossberg and Karshon's description of Bott towers [GK94] as well as [CM12]. We refer to the recent book [BP15] and references therein for further developments.

1.1. Bott towers and their cohomology rings.

Definition 1.1. Given $n \in \mathbb{N}$, we construct complex manifolds M_k for $k \in \{0, 1, \ldots, n\}$ inductively as follows. Let M_0 be a point pt, and for $k \geq 1$, assume M_{k-1} is already defined and choose a holomorphic line bundle $\mathcal{L}_k$ on M_{k-1}. Then M_k is the compact complex manifold arising as the total space of the $\mathbb{CP}^1$ bundle $\pi_k \colon \mathbb{P}(\mathbb{1} \oplus \mathcal{L}_k) \to M_{k-1}$.

We call M_k the *stage* k *Bott manifold* of the *Bott tower of height* n:

$$M_n \xrightarrow{\pi_n} M_{n-1} \xrightarrow{\pi_{n-1}} \cdots M_2 \xrightarrow{\pi_2} M_1 = \mathbb{CP}^1 \xrightarrow{\pi_1} pt.$$

At each stage we have *zero* and *infinity sections* $\sigma_k^0 \colon M_{k-1} \to M_k$ and $\sigma_k^\infty \colon M_{k-1} \to M_k$ which respectively identify M_{k-1} with $\mathbb{P}(\mathbb{1} \oplus 0)$ and $\mathbb{P}(0 \oplus \mathcal{L}_k)$. We consider these to be part of the structure of the Bott tower $(M_k, \pi_k, \sigma_k^0, \sigma_k^\infty)_{k=1}^n$.

Notice that the stage 2 Bott manifolds are nothing but the Hirzebruch surfaces $\mathcal{H}_a := \mathbb{P}(\mathbb{1} \oplus \mathcal{O}(a)) \to \mathbb{CP}^1$.

Definition 1.2. A *Bott tower isomorphism* between Bott towers $(M_k, \pi_k, \sigma_k^0, \sigma_k^\infty)_{k=1}^n$ and $(\tilde{M}_k, \tilde{\pi}_k, \tilde{\sigma}_k^0, \tilde{\sigma}_k^\infty)_{k=1}^n$ is a sequence of biholomorphisms $F_k\colon M_k \to \tilde{M}_k$ (for $k \in \{0, 1, \dots, n\}$) which intertwine the maps $\pi_k, \sigma_k^0, \sigma_k^\infty$ with the maps $\tilde{\pi}_k, \tilde{\sigma}_k^0, \tilde{\sigma}_k^\infty$.

Isomorphism of Bott towers of height n is thus a stronger notion than biholomorphism between the corresponding stage n Bott manifolds.

Remark 1.3. We can consider the more general case where we have a projectivization

$$\mathbb{P}(E_n) \to \mathbb{P}(E_{n-1}) \to \cdots \to \mathbb{P}(E_2) \to \mathbb{P}(E_1) \to pt$$

where the rank 2 bundles E_j do not necesarily split as they do in a Bott tower. Of course, E_2 splits by a famous theorem of Grothendieck. But this fails at the next stage since a rank 2 bundle over a Hirzebruch surface need not split as the first Hirzebruch surface $\mathcal{H}_1$ shows. It would be interesting to investigate this further. However the methods of the present paper rely in an essential way on the toric geometry of the split case, which we introduce in the next section.

As the total space of a $\mathbb{CP}^1$ bundle $\pi_k\colon \mathbb{P}(\mathbb{1} \oplus \mathcal{L}_k) \to M_{k-1}$, M_k has a fiberwise dual tautological bundle $\mathcal{O}(1)_{\mathbb{1}\oplus\mathcal{L}_k}$ whose first Chern class $c_1(\mathcal{O}(1)_{\mathbb{1}\oplus\mathcal{L}_k})$ is the Poincaré dual $PD(D_k^\infty)$ to the infinity section $D_k^\infty := \sigma_k^\infty(M_{k-1})$.

The vertical bundle VM_k of $M_k \to M_{k-1}$ has first Chern class $PD(D_k^0 + D_k^\infty)$, as the generator of the fiberwise $\mathbb{C}^\times$ action vanishes there. Since $VM_k = \mathcal{O}(2)_{\mathbb{1}\oplus\mathcal{L}_k} \otimes \pi_k^*\mathcal{L}_k$, it follows that $\pi_k^*\mathcal{L}_k$ has first Chern class $PD(D_k^0 - D_k^\infty)$, and we let α_k be the pullback of this class to M_n. We let x_k be the pullback of $PD(D_k^\infty)$ to M_n; thus $y_k := x_k + \alpha_k$ is the pullback of $PD(D_k^0)$ to M_n.

Proposition 1.4 ([CM12])**.** *The map sending x_j to $X_j + \mathcal{I}$ induces a ring isomorphism*

$$H^*(M_n, \mathbb{Z}) \cong \mathbb{Z}[X_1, X_2, \dots, X_n]/\mathcal{I} \tag{1}$$

where $\mathcal{I}$ is the ideal generated by

$$X_k^2 + \tilde{\alpha}_k(X_1, \dots, X_{k-1})X_k \quad \textit{for} \quad k \in \{1, \dots, n\}$$

and $\tilde{\alpha}_k$ is the (unique) linear polynomial with $\tilde{\alpha}_k(x_1, \dots, x_{k-1}) = \alpha_k$.

Proof. Since 1 and $PD(D_k^\infty)$ restrict to a basis for the cohomology of any $\mathbb{CP}^1$ fiber of M_k over M_{k-1}, the Leray–Hirsch Theorem implies that the cohomology ring $H^*(M_k, \mathbb{Z})$ is a free module over $H^*(M_{k-1}, \mathbb{Z})$ with generators 1 and $PD(D_k^\infty)$. The result follows by induction, using the fact that for all $k \in \{1, \dots, n\}$, $x_k y_k = 0$ in $H^*(M_n, \mathbb{Z})$. □

Note that the cohomology ring of M_n is filtered by pullbacks of the cohomology rings of M_k for $0 \le k \le n$.

1.2. **The quotient construction and examples.** A stage n Bott manifold M_n can be written as a quotient of n copies of $\mathbb{C}_*^2 := \mathbb{C}^2 \setminus \{0\}$ by a complex n-torus $(\mathbb{C}^\times)^n$. To see this, consider the action of $(t_i)_{i=1}^n \in (\mathbb{C}^\times)^n$ on $(z_j, w_j)_{j=1}^n \in (\mathbb{C}_*^2)^n$ by

$$(t_i)_{i=1}^n\colon (z_j, w_j)_{j=1}^n \mapsto \Big(t_j z_j, \Big(\prod_{i=1}^n t_i^{A_j^i}\Big) w_j\Big)_{j=1}^n, \tag{2}$$

where A is a lower triangular unipotent integer-valued matrix

$$A = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ A^1_2 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A^1_{n-1} & A^2_{n-1} & \cdots & 1 & 0 \\ A^1_n & A^2_n & \cdots & A^{n-1}_n & 1 \end{pmatrix}, \qquad A^i_j \in \mathbb{Z}. \tag{3}$$

Since the induced action of $(\mathbb{R}^+)^n$ is transverse to $(S^3)^n$, where S^3 is the unit sphere in $\mathbb{C}^2_*$, the orbits of this action are in bijection with orbits of the induced free $(S^1)^n$ action on $(S^3)^n$, and the geometric quotient is a compact complex n-manifold $M_n(A)$.

Proposition 1.5 ([GK94, Ish12])**.** *There is a bijection between isomorphism classes of height n Bott towers and $\mathbb{Z}^{n(n-1)/2}$, i.e., matrices A as in* (3)*: $M_n(A)$ is the unique Bott tower for which the pullback of $c_1(\mathcal{L}_k)$ to the total space is $\alpha_k = \sum_{j=1}^{k-1} A^j_k x_j$ for $k \in \{1, \dots, n\}$. In particular the isomorphism class of a height n Bott tower is determined by its filtered cohomology ring.*

Proof. Observe that $M_k(A)$ be the quotient of $(\mathbb{C}^2_*)^k$ by the action (2) with n replaced by k for $0 \le k \le n-1$; then $M_k(A) = \mathbb{P}(\mathbb{1} \oplus \mathcal{L}_k) \to M_{k-1}(A)$ for some line bundle $\mathcal{L}_k$. As shown in [GK94], a holomorphic line bundle on $M_k(A)$ is determined by its first Chern class, and it follows inductively that $M_n(A)$ is a Bott tower with the given Chern classes. Now the matrix A is determined by the filtered cohomology ring of the Bott tower and the result follows [Ish12]. □

In this correspondence, it follows [CR05, Prop. 3.5] that the height k Bott tower associated to the principal $k \times k$ submatrix of A given by removing the first j rows and columns, and the last $n-j-k$ rows and columns is the fiber of $M_{j+k}(A)$ over $M_j(A)$.

We turn now to examples; it is useful to organize these using the following notions.

Definition 1.6. The (*holomorphic*) *twist* of a Bott tower $M_n(A)$ is the number $t \in \{0, \dots, n-1\}$ of holomorphically nontrivial $\mathbb{CP}^1$ bundles in the tower, or equivalently the number of nonzero rows in $A - \mathbb{1}_n$. Dually, we refer to the number $t' \in \{0, \dots, n-1\}$ of nonzero columns in $A - \mathbb{1}_n$ as the (*holomorphic*) *cotwist*. (The kth column of $A - \mathbb{1}_n$ is zero if and only if $M_n(A) \to M_k(A)$ is a pullback from $M_{k-1}(A)$.)

The only Bott manifold M_n of twist (or cotwist) 0 is $(\mathbb{CP}^1)^n$ with cohomology ring

$$H^*(M_n, \mathbb{Z}) = \mathbb{Z}[X_1, \dots, X_n]/\big(X_1^2, \dots, X_n^2\big). \tag{4}$$

We now consider some examples of twist 1 and 2.

Example 1.1. For $\mathbf{k} = (k_1, \dots, k_N) \in \mathbb{Z}^N$, let $M_{N+1}(\mathbf{k})$ denote the Bott tower $M_{N+1}(A)$ with

$$A = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ k_1 & k_2 & \cdots & k_N & 1 \end{pmatrix}, \tag{5}$$

which is the $\mathbb{CP}^1$ bundle $\mathbb{P}(\mathbb{1} \oplus \mathcal{O}(k_1, \dots, k_N)) \to (\mathbb{CP}^1)^N$. Hence it has twist 1 unless $\mathbf{k} = 0$, and its cohomology ring is

$$\begin{aligned} H^*(M_{N+1}(\mathbf{k}), \mathbb{Z}) &= \mathbb{Z}[X_1, \dots, X_{N+1}]/\mathcal{I} \\ \mathcal{I} &= \big(X_1^2, \dots, X_N^2, X_{N+1}(X_{N+1} + k_1 X_1 + \cdots + k_N X_N)\big). \end{aligned} \tag{6}$$

The case $N = 1$ and $k_1 = a$ is the Hirzebruch surface $\mathcal{H}_a = M_n(A)$ with $A = \left(\begin{smallmatrix} 1 & 0 \\ a & 1 \end{smallmatrix}\right)$.

Example 1.2. Similarly for $\mathbf{l} = (l_1, \dots, l_{N-1}) \in \mathbb{Z}^{N-1}$ and $\mathbf{k} = (k_1, \dots, k_N) \in \mathbb{Z}^N$, let $M_{N+1}(\mathbf{l}, \mathbf{k})$ denote the Bott tower $M_{N+1}(A)$ with

$$A = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ l_1 & l_2 & \cdots & 1 & 0 \\ k_1 & k_2 & \cdots & k_N & 1 \end{pmatrix}, \tag{7}$$

This is a fiber bundle over $(\mathbb{CP}^1)^{N-1}$ whose fiber is the Hirzebruch surface $\mathcal{H}_{k_N}$, and has twist 2 unless $\mathbf{k} = 0$ or $\mathbf{l} = 0$. Its cohomology ring is given by

$$\begin{aligned} H^*(M_{N+1}(\mathbf{k}, \mathbf{l}), \mathbb{Z}) &= \mathbb{Z}[X_1, \dots, X_{N+1}]/\mathcal{I} \\ \mathcal{I} &= \big(X_1^2, \dots, X_{N-1}^2, X_N(X_N + l_1 X_1 + \cdots + l_{N-1} X_{N-1}), \\ &\qquad X_{N+1}(X_{N+1} + k_1 X_1 + \cdots + k_N X_N)\big). \end{aligned} \tag{8}$$

Example 1.3. For stage 3 Bott towers, we follow the notation of [CMS10] and let $M_3(a, b, c)$ denote the Bott tower $M_3(A)$ with

$$A = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix}. \tag{9}$$

This is the total space of $\mathbb{CP}^1$ bundle over the Hirzebruch surface $\mathcal{H}_a$, and also a bundle of Hirzebruch surfaces over $\mathbb{CP}^1$ with fiber $\mathcal{H}_c$. Thus it has twist ≤ 1 if $a = 0$ and cotwist ≤ 1 if $c = 0$. Its cohomology ring is

$$H^*(M_3, \mathbb{Z}) = \mathbb{Z}[X_1, X_2, X_3]/\big(X_1^2, X_2(aX_1 + X_2), X_3(bX_1 + cX_2 + X_3)\big), \tag{10}$$

which, as a $\mathbb{Z}$-module, is freely generated by $1, x_1, x_2, x_3, x_1x_2, x_2x_3, x_3x_1$ and $x_1x_2x_3$ (where x_i is the image of X_i in $H^2(M_3, \mathbb{Z})$).

1.3. **Toric description of Bott manifolds.** The quotient construction shows that any Bott manifold is toric. For general descriptions of toric geometry we refer to [BP15, CLS11, Dan78, Del88, LT97], but we summarize the main ideas here in order to declare our conventions.

Definition 1.7. For a finite set $\mathcal{S}$, let $\mathbb{Z}_\mathcal{S}$ and $\mathbb{C}_\mathcal{S} = \mathbb{Z}_\mathcal{S} \otimes_Z \mathbb{C}$ be respectively the free abelian group and complex vector space with basis $e_\rho : \rho \in \mathcal{S}$, and let the complex torus $\mathbb{C}_\mathcal{S}^\times := \mathbb{Z}_\mathcal{S} \otimes_Z \mathbb{C}^\times$ act diagonally on $\mathbb{C}_\mathcal{S}$ in the natural way, i.e., $(t_\rho)_{\rho\in\mathcal{S}} \colon (z_\rho)_{\rho\in\mathcal{S}} \mapsto (t_\rho z_\rho)_{\rho\in\mathcal{S}}$. A complex n-manifold (or orbifold) M is *toric* if it is a quotient of a $\mathbb{C}_\mathcal{S}^\times$-invariant open subset of $\mathbb{C}_\mathcal{S}$ by a subgroup G^c of $\mathbb{C}_\mathcal{S}^\times$ for some $\mathcal{S}$.

It follows that the quotient torus $\mathbb{T}^c \cong \mathbb{C}_{\mathcal{S}}^\times/G^c$ acts on M with an open orbit. To describe such M up to equivariant biholomorphism, it is convenient to fix $\mathbb{T}^c = \Lambda \otimes_{\mathbb{Z}} \mathbb{C}^\times$, where Λ is a free abelian group of rank n (the lattice of circle subgroups of $\mathbb{T}^c$), and determine G^c as the kernel of group homomorphism $\mathbb{C}_{\mathcal{S}}^\times \to \mathbb{T}^c$. Any such homomorphism is determined by a homomorphism $\mathbf{u}\colon \mathbb{Z}_{\mathcal{S}} \to \Lambda$, hence by the images $u_\rho \in \Lambda$ of the basis vectors e_ρ for all $\rho \in \mathcal{S}$. The $\mathbb{C}_{\mathcal{S}}^\times$-orbits in $\mathbb{C}_{\mathcal{S}}$ may be parametrized by subsets of $\mathcal{S}$, where $R \subseteq \mathcal{S}$ corresponds to the orbit $\mathbb{C}_{\mathcal{S},R}$ of $\sum_{\rho \notin R} e_\rho$ (thus $\mathbb{C}_{\mathcal{S},\varnothing}$ is the open orbit).

Hence any $\mathbb{C}_{\mathcal{S}}^\times$-invariant subset of $\mathbb{C}_{\mathcal{S}}$ has the form $\mathbb{C}_{\mathcal{S},\Phi} := \bigcup_{R\in\Phi} \mathbb{C}_{\mathcal{S},R}$ for some subset Φ of the power set $P(\mathcal{S})$, and it is open if and only if Φ is a *simplicial set*, i.e., $R \in \Phi$ and $R' \subseteq R$ implies that $R' \in \Phi$. We also assume all singletons $\{\rho\}$ are in Φ.

Any $R \subseteq \mathcal{S}$ defines a *cone* σ_R in $\mathfrak{t} := \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ as the convex hull of $\{u_\rho \,|\, \rho \in R\}$. We require that the cones $\sigma_R : R \in \Phi$ are distinct, strictly convex (they contain no nontrivial linear subspace) and simplicial ($\dim \sigma_R$ is the cardinality of R), are closed under intersections, and have union $\mathfrak{t}$. Then $\Sigma := \{\sigma_R : R \in \Phi\}$ is a *complete simplicial fan*, and there is a bijection between the cones in the fan and the orbits of $\mathbb{C}_{\mathcal{S}}^\times$ in $\mathbb{C}_{\mathcal{S},\Phi}$, hence the orbits of $\mathbb{T}^c$ in M. Let Σ_r denote the r-dimensional cones in Σ; it is convenient to identify $\mathcal{S}$ with Σ_1, the set of 1-dimensional cones, or rays, in Σ; then $u_\rho \in \rho \cap \Lambda$, and is a multiple m_ρ of the unique primitive vector of Λ in ρ, where $2\pi/m_\rho$ is the cone angle of the orbifold singularity along the corresponding $\mathbb{T}^c$-orbit in M, whose closure D_ρ is a $\mathbb{T}^c$-invariant divisor in M. We refer to $u_\rho\colon \rho \in \mathcal{S}$ as the *normals* of the fan: if M is smooth (or has no orbifold singularities in codimension one), the fan Σ determines $(\mathcal{S}, \mathbf{u}, \Phi)$, hence M; in general the normals are extra data.

Applying these methods to the Bott tower $M_n(A)$, we now see that the role of the matrix A in (2)–(3) is to define an inclusion of lattices $\binom{I_n}{A}\colon \mathbb{Z}^n \hookrightarrow \mathbb{Z}^{2n}$, where I_n is the $n\times n$ identity matrix, and hence real and complex subtori $(S^1)^n \hookrightarrow (S^1)^{2n}$ and $(\mathbb{C}^\times)^n \hookrightarrow (\mathbb{C}^\times)^{2n}$ respectively of the standard $2n$-torus and its complexification acting on $\mathbb{C}^{2n}$. We thus identify $\mathbb{Z}_{\mathcal{S}}$ with $\mathbb{Z}^{2n}$, although it is convenient to take $\mathcal{S} = \{1, \dots, n, 1', \dots, n'\}$ to be the disjoint union of two copies of $\{1, \dots, n\}$ and define $w_j := z_{j'}$, so that $\mathbb{C}_{\mathcal{S}} \cong \mathbb{C}^{2n}$ with coordinates $(z_1, \dots, z_n, w_1, \dots, w_n)$.

Now the image of the inclusion $\binom{I_n}{A}$ is the kernel of the map $(-A \;\; I_n)\colon \mathbb{Z}^{2n} \to \mathbb{Z}^n$. The fan Σ therefore has normals $u_1, \dots, u_n, v_1, \dots, v_n \in \Lambda$ with $u_j = -\sum_{i=1}^n A_i^j v_i$ and $v_j = u_{j'}$; however $v_1, \dots, v_n$ is a $\mathbb{Z}$-basis for Λ, which we may use to identify Λ with $\mathbb{Z}^n$. In particular the quotient torus $\mathbb{T}^c \cong (\mathbb{C}^\times)^n$ acts on $M_n(A)$ and the images of the coordinate hyperplanes $z_k = 0$ or $w_k = 0$ are $\mathbb{T}^c$-invariant divisors in $M_n(A)$.

We can describe these invariant divisors by noting that the quotient of $\{(z_i, w_i)_{i=1}^n \in (S^3)^n : z_j = 0\}$ by the jth circle in $(S^1)^n$ is (since $|w_j| = 1$) $(S^3)^{n-1}$ with the jth 3-sphere removed, and similarly for $\{(z_i, w_i)_{i=1}^n \in (S^3)^n : w_j = 0\}$. The image of $z_j = 0$ or $w_j = 0$ in $M_n(A)$ is thus the height $n-1$ Bott tower obtained by removing the jth row and jth column of A. In terms of the Bott tower structure, the $\mathbb{T}^c$ action on $M_n(A)$ is a lift of the the fiberwise $\mathbb{C}^\times$ actions acting on each stage $M_k(A) \to M_{k-1}(A)$, and the invariant divisors in $M_n(A)$ given by $z_k = 0$ and $w_k = 0$ are the inverse images $\hat{D}_k^\infty$ and $\hat{D}_k^0$ in $M_n(A)$ of D_k^∞ and D_k^0, whose homology classes are Poincaré dual to x_k and

$$y_k = x_k + \alpha_k = \sum_{j=1}^n A_k^j x_j. \tag{11}$$

It will be convenient later to write this in vector notation $\mathbf{y} = A\mathbf{x}$.

There are $2n$ such divisors coming in opposite pairs $(\hat{D}_k^0, \hat{D}_k^\infty) : k \in \{1, \dots, n\}$, labelled by the Bott tower structure, and only opposite pairs have empty intersection. The divisors thus have the structure of an *n-cross* or *cross-polytope* (the dual of an n-cube) and the cones of the fan are therefore cones over the faces of an n-cross in $\mathfrak{t}$ with vertex set $\mathcal{S}$, the rays of the fan [Civ05].

The symmetry group of an n-cross (or n-cube) is the Coxeter group $BC_n \cong \mathrm{Sym}_n \ltimes \mathbb{Z}_2^n$ acting on $\mathcal{S}$, where the ith generator of the $\mathbb{Z}_2^n$ normal subgroup interchanges i and i' in $\mathcal{S}$, while the quotient group Sym_n permutes the pairs $(1, 1'), \dots, (n, n')$.

Remark 1.8. In [MP08, Theorem 3.4], it is shown that stage n Bott manifolds are the only toric complex manifolds whose fan is (the cone over) an n-cross. A corresponding result does not hold in the orbifold case, even for $n = 2$.

1.4. **Equivalences and the Bott tower groupoid.** It is natural to ask when two height n Bott towers determine the same stage n Bott manifold, i.e., their total spaces are biholomorphic. Since a Bott manifold M_n is a complete toric variety, its biholomorphism group $\mathrm{Aut}(M_n)$ is a linear algebraic group, and so all maximal tori in $\mathrm{Aut}(M_n)$ are conjugate. Hence if M_n and $\tilde{M}_n$ are biholomorphic, there is a ϕ-equivariant biholomorphism $f\colon M_n \to \tilde{M}_n$ for some isomorphism $\phi\colon \mathbb{T}^c \to \tilde{\mathbb{T}}^c$ of the canonical complex tori acting on M_n and $\tilde{M}_n$ (i.e., $f(t \cdot z) = \phi(t) \cdot f(z)$ for all $t \in \mathbb{T}^c$, $z \in M_n$). In other words, M_n and $\tilde{M}_n$ are equivalent as toric complex manifolds.

We also use the fact that any height n Bott tower is isomorphic to $M_n(A)$ for a unique $n \times n$ matrix A, and is toric with respect to a fixed complex n-torus $\mathbb{T}^c \cong (\mathbb{C}^\times)^n$. Thus the set of all isomorphism classes of Bott towers of dimension n can be identified with the set $\mathcal{BT}_0^n$ of unipotent lower triangular $n \times n$ matrices A over $\mathbb{Z}$ (which is in bijection with $\mathbb{Z}^{n(n-1)/2}$) by associating to A the isomorphism class of $M_n(A)$.

Definition 1.9. The n-dimensional *Bott tower groupoid* is the groupoid with object set $\mathcal{BT}_0^n$, where the morphisms from A to A' are the biholomorphisms $M_n(A) \to M_n(A')$. We denote the set of morphisms by $\mathcal{BT}_1^n$.

The orbit of $\mathcal{BT}_1^n$ through A is thus the set of all A' such that $M_n(A')$ that is *equivalent* (i.e., biholomorphic) to $M_n(A)$. The quotient space of orbits $\mathcal{BT}_0^n/\mathcal{BT}_1^n$ is then in bijection with the set $\mathcal{B}_n$ of (biholomorphism classes of) Bott manifolds M_n.

As noted before the definition, we may restrict attention to *ϕ-equivariant* equivalences $f\colon M_n(A) \to M_n(A')$ (for $\phi\colon \mathbb{T}^c \to \mathbb{T}^c$). Such an equivalence pulls back $\mathbb{T}^c$-invariant divisors to $\mathbb{T}^c$-invariant divisors preserving their intersections, i.e., it induces an element of the combinatorial symmetry group $BC_n \cong \mathrm{Sym}_n \ltimes \mathbb{Z}_2^n$. It remains to understand which elements of BC_n are induced by equivalences. For this, we follow Masuda and Panov [MP08] (although these authors work in the smooth context).

We first note that if we only know a Bott manifold M_n as a toric complex manifold, then to write it as a quotient of $(\mathbb{C}_*^2)^n \subset \mathbb{C}_{\mathcal{S}}$ we have to choose a bijection between $\mathcal{S} = \{1, \dots, n, 1', \dots, n'\}$ and the $\mathbb{T}^c$-invariant divisors in M_n (so that all pairs (j, j') correspond to opposite divisors). Now a general n-dimensional subtorus $G^c \leq \mathbb{C}_{\mathcal{S}}^\times$ is defined by the kernel of maximal rank block matrix $(B \;\; C)\colon \mathbb{Z}^{2n} \to \mathbb{Z}^n$ (where B, C are $n \times n$). This kernel is the image of $\left(\begin{smallmatrix} I_n \\ A \end{smallmatrix}\right)\colon \mathbb{Z}^n \hookrightarrow \mathbb{Z}^{2n}$ if $B + CA = 0$, which forces C to be invertible and $A = -C^{-1}B$. The automorphism group $\mathrm{GL}(n, \mathbb{Z})$ acts

on $(B\ \ C)\colon \mathbb{Z}^{2n} \to \mathbb{Z}^n$ by left matrix multiplication. Left multiplication by C^{-1} thus provides the normal form $(-A\ \ I_n)\colon \mathbb{Z}^{2n} \to \mathbb{Z}^n$ for the quotient map that we have used.

In $M_n(A)$, the stabilizers of $\hat{D}_k^0$ and $\hat{D}_k^\infty$ in $\mathbb{T}^c$ are the oriented $\mathbb{C}^\times$ subgroups generated by the images of e_k and $e_{k'}$ under $(-A\ \ I_n)$. Hence the action of BC_n on divisors lifts to the natural action on $\mathbb{Z}^{2n}$ by permutation matrices of the form $\left(\begin{smallmatrix} P & Q \\ Q & P \end{smallmatrix}\right)$. Such a matrix acts on $(-A\ \ I_n)$ by right multiplication to give $(Q - AP\ \ P - AQ)$. It follows that such an element of BC_n is induced by an equivalence of $M_n(A)$ with some $M_n(A')$ if and only if $A' := (P - AQ)^{-1}(AP - Q)$ is lower triangular, and then it is induced by an equivalence $M_n(A) \to M_n(A')$.

Lemma 1.10. *Any element of $\mathbb{Z}_2^n \trianglelefteq BC_n$ is induced by an equivalence $M_n(A) \to M_n(A')$ for some A'. In particular, the Bott towers $M_n(A)$ and $M_n(A^{-1})$ are equivalent.*

Proof. It suffices to show that for each $1 \le k \le n$, the generator τ_k which interchanges the kth column of $-A$ with the kth column of I_n is induced by an equivalence. However, here $Q - AP$ and $P - AQ$ are lower triangular, hence so is $A' = (P - AQ)^{-1}(AP - Q)$.

For the second part, we consider $\tau_1\tau_2\cdots\tau_n$, which sends $(\hat{D}_k^0, \hat{D}_k^\infty)$ to $(\hat{D}_k^\infty, \hat{D}_k^0)$ for each $k \in \{1, \dots, n\}$; hence, $\mathbf{x}' := \tau_1\tau_2\cdots\tau_n(\mathbf{x}) = \mathbf{y}$ and $\mathbf{y}' := \tau_1\tau_2\cdots\tau_n(\mathbf{y}) = \mathbf{x}$. But for the Bott tower $M_n(A)$ we have $\mathbf{y} = A\mathbf{x}$, whereas, for $M_n(A^{-1})$ we have $\mathbf{y}' = A^{-1}\mathbf{x}'$. □

The equivalences inducing τ_k are easy to understand as a fiber inversions at the kth stage: if $M_k(A) \cong P(\mathbb{1} \oplus \mathcal{L}_k)$, then $M_k(A') \cong P(\mathbb{1} \oplus \mathcal{L}_k^{-1})$ and in homogeneous coordinates in the fibers, the equivalence sends (z_1, z_2) to (z_1^{-1}, z_2^{-1}), or equivalently, in a local trivialization of $\mathcal{L}_k$, to (z_2, z_1), swapping zero and infinity sections. The equivalences between matrices A, A' induced by the fiber inversion maps can be worked out in principle in terms of minor determinants. Here we only do this for certain special examples.

Before turning to examples, we consider when $\sigma \in \mathrm{Sym}_n$ is induced by an equivalence $M_n(A) \to M_n(A')$. In this case we can take $Q = 0$, so P is the permutation matrix defined by σ, and hence lifts to an equivalence $M_n(A) \to M_n(P^{-1}AP)$ if and only if $P^{-1}AP$ is lower triangular. Since Sym_n is generated by transpositions, the following case is of particular interest.

Lemma 1.11. *A transposition $\sigma = (i\ j)$ with $1 \le i < j \le n$ is induced by an equivalence of $M_n(A) \to M_n(P^{-1}AP)$ for the corresponding permutation matrix P if and only if $A_k^i = 0$ for $i < k \le j$ and $A_j^k = 0$ for $i \le k < j$. In particular, when $i = j - 1$, this holds if and only if $A_j^{j-1} = 0$.*

The proof of this lemma is immediate; we illustrate it in examples below. It has the following consequence, which is one of the main steps in the proof of [CS11, Lemma 3.1].

Proposition 1.12. *A Bott tower $M_n(A)$ with twist $\le t$ is equivalent to a Bott tower $M_n(A')$ such that the first $n - t$ rows of $A' - \mathbb{1}_n$ are identically zero, i.e., $M_{n-t}(A') = (\mathbb{CP}^1)^{n-t}$. Similarly, a Bott tower $M_n(A)$ with cotwist $\le t$ is equivalent to a Bott tower $M_n(A')$ such that the last $n - t$ columns are identically zero, i.e., $M_n(A')$ is a $(\mathbb{CP}^1)^{n-t}$ bundle over $M_t(A')$.*

Proof. If the jth row of $A - \mathbb{1}_n$ is identically zero then iterating Lemma 1.11 shows that the permutation $(1\ 2 \cdots j) = (1\ 2)(2\ 3)\cdots(j-1\ j)$ is induced by an equivalence. Now apply this argument to all such rows. The proof for columns is similar. □

Example 1.4. Proposition 1.12 shows that any Bott tower $M_{N+1}(A)$ of twist ≤ 1 is equivalent to $M_{N+1}(\mathbf{k})$ for some $\mathbf{k} \in \mathbb{Z}^N$. The only fiber inversion that yields a nontrivial equivalence of $M_{N+1}(\mathbf{k})$ is τ_{N+1}, which interchanges $\hat{D}^{\infty}_{N+1}$ and $\hat{D}^{0}_{N+1}$, and is induced by an equivalence between $M_{N+1}(\mathbf{k})$ and $M_{N+1}(-\mathbf{k})$. Also we have $M_{N+1}(\sigma(\mathbf{k})) \sim M_{N+1}(\mathbf{k})$ for any permutation σ that lies in the subgroup $\mathrm{Sym}_N \subset \mathrm{Sym}_{N+1}$ which permutes the first N opposite pairs of invariant divisors. Since $M_{N+1}(\mathbf{k})$ is a non-trivial $\mathbb{CP}^1$ bundle over the product $(\mathbb{CP}^1)^N$, these equivalences are just permuting the N factors of the base.

Example 1.5. Proposition 1.12 also shows that any Bott tower $M_{N+1}(A)$ of twist ≤ 2 is equivalent to $M_{N+1}(\mathbf{l}, \mathbf{k})$ for some $(\mathbf{l}, \mathbf{k}) \in \mathbb{Z}^{N-1} \times \mathbb{Z}^N$. There are now two non-trivial fiber inversion maps, namely, those interchanging $\hat{D}^{\infty}_{N}$ and $\hat{D}^{0}_{N}$ and those interchanging $\hat{D}^{\infty}_{N+1}$ and $\hat{D}^{0}_{N+1}$, giving equivalences

$$M_{N+1}(\mathbf{l}, \mathbf{k}) \sim M_{N+1}(\mathbf{l}, -\mathbf{k}) \sim M_{N+1}(-\mathbf{l}, \mathbf{k}') \sim M_{N+1}(-\mathbf{l}, -\mathbf{k}')$$

where $\mathbf{k}' = (k_1 - l_1 k_N, \ldots, k_{N-1} - l_{N-1} k_N, k_N)$. A twist 2 Bott tower of the form $M_{N+1}(\mathbf{l}, \mathbf{k})$ can be viewed as a fiber bundle over the product $(\mathbb{CP}^1)^{N-1}$ whose fiber is a Hirzebruch surface. So permuting the $N-1$ factors of the base induce equivalences $M_{N+1}(\sigma(\mathbf{l}, \mathbf{k})) \sim M_{N+1}(\mathbf{l}, \mathbf{k})$ where $\sigma \in \mathrm{Sym}_{N-1} \subset \mathrm{Sym}_{N+1}$ where Sym_{N-1} is the subgroup that permutes the first $N-1$ opposite pairs of invariant divisors.

Example 1.6. For the height 3 Bott tower $M_3(A) = M_3(a, b, c)$, the fiber inversion maps give rise to equivalences $\tau_1 \colon M_3(a, b, c) \to M_3(a, b, c)$, $\tau_2 \colon M_3(a, b, c) \to M_3(-a, b - ac, c)$ and $\tau_3 \colon M_3(a, b, c) \to M_3(a, -b, -c)$. Hence we have equivalences:

$$M_3(a, b, c) \sim M_3(a, -b, -c) \sim M_3(-a, b - ac, c) \sim M_3(-a, -(b - ac), -c) = M_3(A^{-1}).$$

The transposition $(1\,3)$ is order reversing, so is induced by an equivalence only when $a = b = c = 0$. The 3-cycles (123) and (132) are induced by equivalences when $b = c = 0$ and $a = b = 0$ respectively. The transposition $(1\,2)$ is induced by an equivalence when $a = 0$ and the transposition $(2\,3)$ is induced by an equivalence when $c = 0$. We conclude that

$$M_3(a, b, 0) \sim M_3(b, a, 0) \quad \text{and} \quad M_3(0, b, c) \sim M_3(0, c, b),$$

hence in particular $M_3(a, 0, 0) \sim M_3(0, a, 0) \sim M_3(0, 0, a)$. These equivalences have straightforward interpretations: $M_3(a, b, 0)$ has cotwist ≤ 1, i.e., is a $\mathbb{CP}^1 \times \mathbb{CP}^1$ over $\mathbb{CP}^1$, and the equivalence interchanges the two factors in the fibers; similarly $M_3(0, b, c)$ has twist ≤ 1, i.e., is a $\mathbb{CP}^1$ bundle over $\mathbb{CP}^1 \times \mathbb{CP}^1$, and the equivalence interchanges the two factors in the base.

Example 1.7. To illustrate Lemma 1.11, observe that transposing the second and fourth rows and columns of a unipotent 4×4 lower triangular matrix A yields

$$A' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ A^1_2 & 1 & A^3_4 & A^2_4 \\ A^1_3 & 0 & 1 & A^2_3 \\ A^1_4 & 0 & 0 & 1 \end{pmatrix}$$

and so the transposition $(2\,4)$ is induced by an equivalence of $M_4(A)$ with $M_4(A')$ if and only if $A^2_3 = A^2_4 = A^3_4 = 0$.

2. The topology of Bott manifolds

2.1. **Topological twist.** There is a close interplay between the topological and biholomorphic theory of Bott manifolds. One example is the following result.

Proposition 2.1 (Choi–Suh [CS11])**.** *In a Bott tower $M_n(A)$, the $\mathbb{CP}^1$ bundle $M_k(A) = \mathbb{P}(\mathbb{1} \oplus \mathcal{L}_k) \to M_{k-1}(A)$ is topologically trivial if and only if $\alpha_k := \sum_{j=1}^{k-1} A_k^j x_j \equiv 0$ mod 2 and $\alpha_k^2 = 0$. In this case $M_n(A)$ is diffeomorphic to a Bott tower $M_n(A')$ such that $M_k(A') = \mathbb{CP}^1 \times M_{k-1}(A)$ is holomorphically trivial over $M_{k-1}(A') = M_{k-1}(A)$.*

Proof. $\mathbb{P}(\mathbb{1} \oplus \mathcal{L}_k)$ is topologically trivial if and only if $(\mathbb{1} \oplus \mathcal{L}_k) \otimes \mathcal{L}$ is topologically trivial for some line bundle $\mathcal{L}$. Choi and Suh [CS11] show that a sum of line bundles over a Bott manifold is topologically trivial if and only if its total Chern class is trivial. If $c_1(\mathcal{L}) = \lambda$ then

$$c(\mathcal{L} \oplus \mathcal{L}_k \otimes \mathcal{L}) = c(\mathcal{L})c(\mathcal{L}_k \otimes \mathcal{L}) = (1+\lambda)(1+\alpha_k+\lambda) = 1 + \alpha_k + 2\lambda + \lambda^2.$$

This is trivial if $\alpha_k = -2\lambda$ and $\lambda^2 = 0$, i.e., $\alpha_k \equiv 0$ mod 2 and $\alpha_k^2 = 0$ (since the cohomology ring has no torsion). For the last part, we pullback $M_n(A) \to M_k(A)$ by the diffeomorphism $M_k(A) \to \mathbb{CP}^1 \times M_{k-1}(A)$ trivializing $M_k(A)$ over $M_{k-1}(A)$. □

The construction in the proof does not affect the topological triviality of the other fibrations in $M_n(A)$, which prompts the following definition.

Definition 2.2. The *topological twist* of a Bott manifold M_n is the minimal (holomorphic) twist among Bott towers $M_n(A)$ diffeomorphic to M_n.

Proposition 2.1 shows that the topological twist of M_n is also the minimal number of topologically nontrivial stages among Bott towers diffeomorphic to M_n. In fact Choi and Suh show [CS11, Theorem 3.2] that the topological twist is the number of topologically nontrivial stages in *any* Bott tower diffeomorphic to M_n (and this is their definition of "twist"). Furthermore, by Proposition 1.12 (cf. [CS11, Lemma 3.1]), a Bott manifold with topological twist t is diffeomorphic to a Bott tower $M_n(A)$ where the first $n-t$ rows of $A - \mathbb{1}_n$ are identically zero, i.e., a holomorphic fiber bundle over $(\mathbb{CP}^1)^{n-t}$ whose fiber is a stage t Bott manifold.

The topological twist of a Bott manifold has implications for its Pontrjagin classes.

Lemma 2.3 ([CMM15, CS11])**.** *Let M_n be a Bott manifold with topological twist $\leq t$.*

(1) *If M_n is diffeomorphic to $M_n(A)$, its total Pontrjagin class $p(M_n)$ is given by*

$$p(M_n) = \prod_{j=n+1-t}^{n} (1+\alpha_j^2); \tag{12}$$

in particular $p_k(M_n)$ vanishes for $k > t$.

(2) $p_k(M_n) = 0$ *if* $k \geq \frac{n}{2}$.

Proof. For (1), the computation of (12) is straightforward [CMM15] and the last part follows by taking $M_n(A)$ to have twist $\leq t$. For (2) we note that the strict inequality $k > \frac{n}{2}$ follows for dimensional reasons. For the equality we set $n = 2m$ and and observe

that the class $p_m(M_{2m})$ is a multiple of $x_1 \cdots x_{2m}$, so when $t = 2m-1$ we have

$$p_m(M_{2m}) = \prod_{j=2}^{2m} \alpha_j^2 = \prod_{j=2}^{2m} \Big(\sum_{i=1}^{j-1} A_j^i x_i \Big)^2$$

which does not contain x_{2m}. □

For later use, we also note that the total Chern class of $M_n(A)$ is

$$c(M_n(A)) = \prod_{j=1}^{n} (1 + x_j + y_j) = \prod_{j=1}^{n} (1 + 2x_j + \alpha_j), \tag{13}$$

cf. [Abc13]. In particular, for the first Chern class we get

$$c_1(M_n(A)) = \sum_{j=1}^{n} (x_j + y_j). \tag{14}$$

2.2. **Cohomological rigidity of Bott manifolds.** In recent years, research on Bott manifolds has centered around the *cohomological rigidity problem* which asks if the integral cohomology ring of a toric complex manifold determines its diffeomorphism (or homeomorphism) type [CMS11, CM12, Cho15, MP08]. This problem is still open even for Bott manifolds, but has an affirmative answer in important special cases, in particular for Bott manifolds with topological twist ≤ 1. This gives a lot of information about the topology of these Bott manifolds.

Bott manifolds with topological twist 0. A Bott manifold M_n has topological twist 0 if and only if it is diffeomorphic to $(\mathbb{CP}^1)^n$. In this case there is the following characterization.

Theorem 2.4 (Masuda–Panov [MP08])**.** *A stage n Bott manifold M_n is diffeomorphic to $(\mathbb{CP}^1)^n$ if and only if $H^*(M_n, \mathbb{Z}) \cong H^*((\mathbb{CP}^1)^n, \mathbb{Z})$ as graded rings. Furthermore, $M_n(A)$ is diffeomorphic to $(\mathbb{CP}^1)^n$ if and only if the matrix A takes the form*[1]

$$A = 2C_n \cdots C_1 - \mathbb{1}_n \tag{15}$$

where C_k is a lower triangular unipotent matrix with at most one non-zero element $C_k^{i_k}$ below the diagonal and that lies in the kth row.

Note that C_1 is the identity matrix. It follows immediately from (15) that all off-diagonal elements of A are multiples of 2 (in accordance with Proposition 2.1).

Example 2.1. As an example we consider stage 3 Bott manifolds with topological twist zero. It follows from (15) (and also Proposition 2.9 below) that they can be represented by the Bott towers $M_3(2a, 2b, 0)$ and $M_3(2a, 2ac, 2c)$. The former has cotwist ≤ 1, hence is a $\mathbb{CP}^1 \times \mathbb{CP}^1$ bundle over $\mathbb{CP}^1$, whereas the latter has twist and cotwist 2. However, the A matrices for $M_3(2a, 2b, 2c)$ with $c(b - ac) \neq 0$ do not satisfy (15); they have non-vanishing first Pontrjagin class.

[1]Comparing notations we note that our A is $-A^t$ in [MP08] with $a_{ij} = -A_j^i$; this transposition also reverses the order of the C_k's.

$\mathbb{Q}$-trivial Bott manifolds. The cohomological rigidity of Bott manifolds with topological twist 0 generalizes to Bott manifolds M_n which are $\mathbb{Q}$-*trivial*, i.e., with $H^*(M_n, \mathbb{Q}) \cong H^*((\mathbb{CP}^1)^n, \mathbb{Q})$ as graded rings [CM12]. A key ingredient in establishing this is the following observation.

Lemma 2.5 ([CM12])**.** *Let $\lambda x_j + u$ be a primitive element of $H^2(M_n, \mathbb{Z})$ with $\lambda \neq 0$ and u in the span of $x_i : i < j$. Then $(\lambda x_j + u)^2 = 0$ if and only if $\alpha_j^2 = 0$ and $2u = \lambda\alpha_j$.*

Proof. By the assumptions on λ and u, $(\lambda x_j + u)^2 = \lambda^2 x_j^2 + 2\lambda u x_j + u^2 = \lambda(2u - \lambda\alpha_j)x_j + u^2 = 0$ if and only if $2u - \lambda\alpha_j = 0$ and $u^2 = 0$. □

Thus the primitive square-zero elements of $H^2(M_n, \mathbb{Z})$ have the form $2x_j + \alpha_j$ or $x_j + \alpha_j/2$ up to sign. It follows easily [CM12] that M_n is $\mathbb{Q}$-trivial if and only if $\alpha_k^2 = 0$ for all $k \in \{1, \ldots, n\}$. Note that $\alpha_1 = 0$ and $\alpha_2^2 = 0$, so stage 2 Bott manifolds, i.e., Hirzebruch surfaces, are always $\mathbb{Q}$-trivial, and it is well known that there are precisely two diffeomorphism types, distinguished by the parity of A_2^1.

More generally, Choi and Masuda [CM12] show that $\mathbb{Q}$-trivial Bott manifolds are distinguished by their integral cohomology rings with $\mathbb{Z}$ coefficients, and in this case there are exactly $P(n)$ diffeomorphism types where $P(n)$ is the number of partitions of n. Further, for any $\mathbb{Q}$-trivial Bott manifold M_n, there is a partition $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 1$ of n such that $M_n \cong M_{(\lambda_1)} \times \cdots \times M_{(\lambda_n)}$ where $M_{(\lambda)}$ is the stage λ Bott manifold associated to the matrix A with $A_k^1 = 1$ for all k and $A_k^j = \delta_k^j$ for $j \geq 2$.

Thus, by the famous formula of Hardy and Ramanujan the number of diffeomorphism types of stage n $\mathbb{Q}$-trivial Bott manifolds grows asymptotically like

$$\frac{\exp(\pi(2/3)^{1/2}\sqrt{n})}{4n\sqrt{3}} \qquad \text{as } n \to \infty.$$

However, for $n \geq 3$, it follows from the formula (12) for the total Pontrjagin class that there are infinitely many diffeomorphism types of stage n Bott manifolds that are not $\mathbb{Q}$-trivial. Nevertheless, the cohomological rigidity problem has an affirmative answer for stage n Bott manifolds with $n \leq 4$ [CMS10, Cho15].

Bott manifolds with topological twist 1. Any stage $N+1$ Bott manifold M_{N+1} with topological twist ≤ 1 is diffeomorphic to $M_{N+1}(\mathbf{k})$ for some $\mathbf{k} = (k_1, \ldots, k_N) \in \mathbb{Z}^N$ as in Example 1.1. These manifolds have each have one nonvanishing Pontrjagin class, viz.

$$p_1(M_{N+1}(\mathbf{k})) = 2\sum_{i<j} k_i k_j x_i x_j. \tag{16}$$

Their first Chern class is given by

$$c_1(M_{N+1}(\mathbf{k})) = \sum_{i=1}^{N}(2 + k_i)x_i + 2x_{N+1},$$

so the second Stiefel–Whitney class is

$$w_2(M_{N+1}(\mathbf{k})) = \sum_{i=1}^{N} k_i x_i \mod 2.$$

In this case the diffeomorphism type is determined by the graded cohomology ring $H^*(\mathbb{P}(\mathbb{1} \oplus \mathcal{O}(k_1, \ldots, k_N)), \mathbb{Z})$.

Theorem 2.6 ([CS11]). *Let $M_{N+1}(\mathbf{k})$ and $M_{N+1}(\mathbf{k}')$ be two Bott towers whose A matrix has the form (5). Then $M_{N+1}(\mathbf{k})$ and $M_{N+1}(\mathbf{k}')$ are diffeomorphic if and only if there is a permutation σ of $\{1,\dots,N\}$ such that $k'_{\sigma(i)} \equiv k_i \mod 2$ for all i and $k'_{\sigma(i)}k'_{\sigma(j)} = \pm k_i k_j$ for all $i \neq j$.*

This theorem characterizes the case that $M_{N+1}(\mathbf{k})$ has topological twist 0, i.e., is diffeomorphic to $(\mathbb{CP}^1)^{N+1}$: this happens precisely when $\mathbf{k}$ has at most one non-vanishing component and it is even; thus $M_{N+1}(\mathbf{k})$ is the product of an even Hirzebruch surface with $(\mathbb{CP}^1)^{N-1}$.

When $N = 2$ we have a stage 3 Bott manifold with twist ≤ 1 which will be treated in Section 2.3. In this case the number of Bott manifolds in a given diffeomorphism type is determined by the prime decomposition of k_1k_2. For $N > 2$, Theorem 2.6 has the following refinement.

Theorem 2.7. *If $N > 2$ and all k_j are nonvanishing, then $M_{N+1}(\mathbf{k})$ is diffeomorphic to $M_{N+1}(\mathbf{k}')$ if and only if there is a permutation σ of $\{1,\dots,N\}$ such that $k'_{\sigma(i)} = \pm k_i$ for all $i = 1,\dots,N$. Moreover, there are generically 2^{N-1} inequivalent such Bott manifolds in each diffeomorphism class.*

Proof. By the theorem $M_{N+1}(\mathbf{k})$ and $M_{N+1}(\mathbf{k}')$ are diffeomorphic if and only if there is a permutation σ such that $\mu_i\mu_j = \pm 1$ for all $i \neq j$, where $\mu_i = k'_{\sigma(i)}/k_i$; hence $\mu_i = \pm 1/\mu_j$ for $i \neq j$. For $N > 2$, we have also $\mu_j = \pm 1/\mu_k$ and $\mu_k = \pm 1/\mu_i$ for $k \neq i, j$, which implies that $\mu_i = \pm 1$, i.e., $k'_{\sigma(i)} = \pm k_i$ for all i. This proves the first part of the theorem.

For the second part we note that the only fiber inversion that yields a nontrivial equivalence is τ_{N+1}, which interchanges $\hat{D}^{\infty}_{N+1}$ and $\hat{D}^{0}_{N+1}$, and is induced by an equivalence between $M_{N+1}(\mathbf{k})$ and $M_{N+1}(-\mathbf{k})$. Now in the diffeomorphism class of $M_{N+1}(\mathbf{k})$, there 2^N choices of sign for the components of $\mathbf{k}$, but the equivalence of $M_{N+1}(\mathbf{k})$ and $M_{N+1}(-\mathbf{k})$ makes half of them equivalent. There are no further equivalences unless $k_i = \pm k_j$ for some $i \neq j$ (in which case some sign choices are identified by transposing the ith and jth factors in the base). Thus there are generically 2^{N-1} inequivalent Bott manifolds in each diffeomorphism class. □

Now we let $\mathcal{A}(\mathbf{k})$ be the set of lower triangular unipotent matrices over $\mathbb{Z}$ such that the Bott tower $M_{N+1}(A_{\mathbf{k}})$ for $A_{\mathbf{k}} \in \mathcal{A}(\mathbf{k})$ is diffeomorphic to $M_{N+1}(\mathbf{k})$. In the degenerate case when some of the k_js vanish we can without loss of generality assume that $k_j \neq 0$ for $j = 1,\dots,m$, but $k_j = 0$ for $j = m+1,\dots,N$. In this case we have a biholomorphism for $m = 2,\dots,N+1$

$$M_{N+1}(A_{\mathbf{k}}) \cong M_m(A_{\tilde{\mathbf{k}}}) \times \overbrace{\mathbb{CP}^1 \times \cdots \times \mathbb{CP}^1}^{N+1-m} \tag{17}$$

where none of the components $k_1,\dots,k_{m-1}$ of $\tilde{\mathbf{k}}$ vanish.

Bott manifolds with topological twist 2. A Bott manifold with topological twist ≤ 2 is diffeomorphic to a Bott tower of the form $M_{N+1}(\mathbf{l},\mathbf{k})$ as in Example 1.2. Less is known about the topology in this case; their only non-vanishing α_j are for $j = N, N+1$ with

$$\alpha_N^2 = 2\sum_{i<j}^{N-1} l_i l_j x_i x_j, \quad \alpha_{N+1}^2 = 2\sum_{i<j}^{N-1} k_i k_j x_i x_j + k_N \sum_{i=1}^{N-1}(2k_i - k_N l_i)x_i x_N.$$

The total Pontrjagin class $p(M_{N+1}(\mathbf{l},\mathbf{k}))$ is a diffeomorphism invariant and from Lemma 2.3 there are at most only 2 non-vanishing classes:

$$p_1(M_{N+1}(\mathbf{l},\mathbf{k})) = \alpha_N^2 + \alpha_{N+1}^2, \qquad p_2(M_{N+1}(\mathbf{l},\mathbf{k})) = \alpha_N^2\alpha_{N+1}^2. \tag{18}$$

Notice also by Lemma 2.3 that for $N = 3$ we have $p_2(M_4(\mathbf{l},\mathbf{k})) = 0$.

We also have

$$c_1(M_{N+1}(\mathbf{l},\mathbf{k})) = \sum_{i=1}^{N-1}(2 + l_i + k_i)x_i + (2 + k_N)x_N + 2x_{N+1}$$

from which we obtain the second Stiefel–Whitney class

$$w_2(M_{N+1}(\mathbf{l},\mathbf{k})) = \sum_{i=1}^{N-1}(l_i + k_i)x_i + k_N x_N \mod 2.$$

This implies that $w_2 = 0$ if and only if k_N is even and l_i, k_i have the same parity for all $i = 1, \dots, N-1$.

2.3. **Topological classification of stage 3 Bott manifolds.** Let us consider in detail the topology of height 3 Bott towers $M_3(a,b,c)$. From (12) we have

$$p_1(M_3) = \alpha_3^2 = c(2b - ac)x_1x_2, \tag{19}$$

while from (14) we have

$$c_1(M_3) = (2 + a + b)x_1 + (2 + c)x_2 + 2x_3. \tag{20}$$

The mod 2 reduction of c_1 is the second Stiefel–Whitney class w_2 so we see that

$$w_2(M_3) \equiv (a + b)x_1 + cx_2 \mod 2. \tag{21}$$

Lemma 2.8. *The Bott tower $M_3(a,b,c)$ is $\mathbb{Q}$-trivial if and only if $p_1(M_3) = 0$ if and only if $c(2b - ac) = 0$. Furthermore $w_2(M_3) = 0$ if and only if c and $a + b$ are even.*

By Lemma 2.5, the primitive square-zero elements of $H^2(M_3,\mathbb{Z})$ are, up to sign,

$$\beta_1 = x_1, \quad \beta_2 = \begin{cases} x_2 + \frac{1}{2}ax_1 & a \text{ even} \\ 2x_2 + ax_1 & a \text{ odd,} \end{cases} \quad \beta_3 = \begin{cases} x_3 + \frac{1}{2}bx_1 + \frac{1}{2}cx_2 & b, c \text{ even} \\ 2x_3 + bx_1 + cx_2 & \text{otherwise,} \end{cases} \tag{22}$$

with β_3 only occuring in the $\mathbb{Q}$-trivial case. The topological twist 0 case is characterized as follows.

Proposition 2.9. *A Bott tower $M_3(a,b,c)$ is diffeomorphic to $(\mathbb{CP}^1)^3$ if and only if it is $\mathbb{Q}$-trivial (i.e., $c(2b - ac) = 0$) and a, b, c are all even.*

Proof. Clearly if the integral cohomology ring of M_3 is isomorphic to that of $(\mathbb{CP}^1)^3$ then M_3 is $\mathbb{Q}$-trivial, and since $\beta_1\beta_2\beta_3$ is primitive, a, b and c are all even. Conversely, these conditions imply that $\beta_1, \beta_2, \beta_3$ have square zero and span $H^2(M,\mathbb{Z})$, so that they generate $H^*(M,\mathbb{Z})$. Let ξ_1, ξ_2, ξ_3 be the lifts of $\beta_1, \beta_2, \beta_3$ to $\mathbb{Z}[x_1, x_2, x_3]$ using the explicit expressions (22). It is straightforward to check that ξ_1^2, ξ_2^2 and ξ_3^2 generate the ideal $\mathcal{I}$. Hence $H^*(M,\mathbb{Z}) \cong \mathbb{Z}[\xi_1, \xi_2, \xi_3]/(\xi_1^2, \xi_2^2, \xi_3^2)$ and the result follows by Theorem 2.4. □

Remark 2.10. Note that Proposition 2.9 in particular implies that the total space of $\mathbb{P}(\mathbb{1} \oplus \mathcal{O}(k_1, k_2)) \to \mathbb{CP}^1 \times \mathbb{CP}^1$ with $k_1k_2 \neq 0$ is never diffeomorphic to $(\mathbb{CP}^1)^3$.

From [CM12] we know that for $\mathbb{Q}$-trivial stage 3 Bott manifolds there are precisely three diffeomorphism types. The above proposition shows that $M_3(a,b,c) \cong M^3_{(1)} = (\mathbb{CP}^1)^3$ if and only if a,b,c are all even. The other two diffeomorphism types can be distinguished by the second Stiefel–Whitney class $w_2(M_3)$. If a,b are odd and c is even then $w_2(M_3) = 0$ and M_3 is diffeomorphic to $M_{(3)}$. Otherwise, $w_2(M_3) \neq 0$, M_3 is diffeomorphic to $M_{(1)} \times M_{(2)}$, and the equation $c(2b - ac) = 0$ implies that either c is even and $a + b$ is odd, with $w_2(M_3) \equiv \beta_2 \equiv x_1$, or c is odd and a is even, with

$$w_2(M_3) \equiv \beta_3 \equiv \begin{cases} x_2 & b \text{ even} \\ x_1 + x_2 & b \text{ odd.} \end{cases}$$

Thus the isomorphism of cohomology rings between $\mathbb{Q}$-trivial height 3 Bott towers $M_3(a,b,c)$ and $M_3(a',b',c')$ with $w_2 \neq 0$ maps $\beta_j \to \beta'_j$ for all j if a and a' have the same parity; otherwise it maps β_2 to β'_3 and β_3 to β'_2.

Proposition 2.11. *Let $M_3(a,b,c)$ and $M_3(a',b',c')$ be Bott towers which are not $\mathbb{Q}$-trivial. Then $M_3(a,b,c)$ and $M_3(a',b',c')$ are diffeomorphic if and only if $c(2b - ac) = \pm c'(2b' - a'c')$, (a,a') have the same parity, and:*

- *if (a,a') are both even, then $((1+b)(1+c),(1+b')(1+c'))$ have the same parity;*
- *if (c,c') are both even, then (b,b') have the same parity.*

Proof. If $M_3(a,b,c)$ and $M_3(a',b',c')$ are diffeomorphic, there is an isomorphism ψ of their integral cohomology rings intertwining their Pontrjagin and Stieffel–Whitney classes. Since x_1x_2 and $x'_1x'_2$ are primitive, this implies $c(2b - ac) = \pm c'(2b' - a'c')$. Since the two Bott towers are not $\mathbb{Q}$-trivial, ψ induces a bijection between $\{\pm\beta_1, \pm\beta_2\}$ and $\{\pm\beta'_1, \pm\beta'_2\}$ and hence $\beta_1\beta_2$ is mapped to $\pm\beta'_1\beta'_2$. Thus (a,a') have the same parity, and the last two conditions follow by considering whether w_2 vanishes or not.

Conversely, given the assumptions on a,b,c and a',b',c', it suffices by [CMS10] to show that $M_3(a,b,c)$ and $M_3(a',b',c')$ have isomorphic integral cohomology rings. Replacing x_1 by $-x_1$, we see that $M_3(a,b,c)$ and $M(-a,-b,c)$ have isomorphic cohomology, so we may assume that $c(2b - ac) = c'(2b' - a'c')$. Replacing x_2 by $x_2 + \lambda x_1$ for $\lambda \in \mathbb{Z}$, we see that $M_3(a,b,c)$ and $M_3(a + 2\lambda, b + c\lambda, c)$ have isomorphic cohomology, so we may assume $a = a'$. Thus $2bc - 2b'c' = a(c + c')(c' - c)$. If a is odd then c and c' have the same parity and hence $bc \equiv b'c' \bmod 2$, so that b and b' have the same parity (by assumption if c and c' are even). If $a = 2\tilde{a}$ then $bc - b'c' = \tilde{a}(c + c')(c' - c)$ hence $(1 + b)(1 + c) - (1 + b')(1 + c') + (b' - b) = (\tilde{a}(c + c') - 1)(c' - c)$. Since the first two terms have the same parity, (b,b') have the same parity if and only if (c,c') do. In the opposite parity case, $\tilde{a}$ must be even, and it follows that (b,c') and (c,b') have the same parity. Now replacing x_1 by $x_2 + (a/2)x_1$ and x_2 by $(1 - a^2/4)x_1 - (a/2)x_2$, we see that $M_3(a,b,c)$ and $M_3(a, c + (a/2)b - (a^2/4)c, b - (a/2)c)$ have isomorphic cohomology. Hence we may assume that (b,b') have the same parity and (c,c') have the same parity (as in the case a odd).

Finally, replacing x_3 by $x_3 - \frac{1}{2}(b - b')x_1 - \frac{1}{2}(c - c')x_2$, we conclude that $M_3(a,b,c)$ and $M_3(a',b',c')$ have isomorphic integral cohomology. □

The cotwist ≤ 1 case, when M_3 is a $\mathbb{CP}^1 \times \mathbb{CP}^1$ bundle over $\mathbb{CP}^1$, is $c = 0$, and the ($\mathbb{Q}$-trivial) cohomology ring reduces to

$$H^*(M_3, \mathbb{Z}) = \mathbb{Z}[x_1, x_2, x_3]/\big(x_1^2, x_2(ax_1 + x_2), x_3(bx_1 + x_3)\big),$$

whereas the twist ≤ 1 case, when M_3 is a $\mathbb{CP}^1$ bundle over the product $\mathbb{CP}^1 \times \mathbb{CP}^1$, is $a = 0$, and the cohomology ring reduces to

$$H^*(M_3(0,b,c),\mathbb{Z}) = \mathbb{Z}[x_1,x_2,x_3]/\big(x_1^2, x_2^2, x_3(bx_1+cx_2+x_3)\big). \tag{23}$$

In the twist ≤ 1 case, the parameters b, c are the bidegree of the line bundle $\mathcal{O}(b,c)$ over $\mathbb{CP}^1 \times \mathbb{CP}^1$, so $M_3(0,b,c)$ can be realized as the projectivization $\mathbb{P}(\mathbb{1} \oplus \mathcal{O}(b,c))$, which fits into the general admissible construction of [ACGT08a]. This case includes the Kähler–Einstein examples of Koiso and Sakane [KS86] with $(b,c) = (1,-1)$ as well as other extremal and CSC metrics in [Hwa94, Gua95] as briefly discussed above.

More precisely, $p_1(M_3)$ is the multiple $p = 2bc$ of the primitive integral class $x_1x_2 = \frac{1}{2}c_1(\mathcal{O}(1,1))^2$ (pulled back to M_3) and $w_2(M_3) \equiv bx_1 + cx_2$ mod 2. Hence the number of Bott manifolds with twist ≤ 1 in a fixed diffeomorphism type is the number of factorizations bc of p with fixed parity of $(1+b)(1+c)$ (which is odd if and only if b, c are both even). This may be determined easily from the prime decomposition of p.

Example 2.2. For example consider $bc = \pm 24$. Proposition 2.11 gives two distinct diffeomorphism types with diffeomorphisms $M_3(0,24,1) \cong M_3(0,24,-1) \cong M_3(0,8,3) \cong M_(0,8,-3)$ and $M_3(0,12,2) \cong M_3(0,12,-2) \cong M_3(0,6,4) \cong M_3(0,6,4)$. The first set has $w_2 \neq 0$ while the second has vanishing w_2, so the two sets are distinct even as homotopy types. It is interesting to note that when bc is negative, we have, as mentioned previously, CSC Kähler metrics [Hwa94].

3. The complex viewpoint

3.1. **The automorphism group.** The isotropy subgroup $\mathrm{Aut}(M_n(A))$ of $\mathcal{BT}_1^n$ of the Bott tower $M_n(A) \in \mathcal{BT}_0^n$ is by definition the automorphism group $\mathrm{Aut}(M_n)$ of the underlying Bott manifold M_n. We let $\mathrm{Aut}_0(M_n(A))$ denote its identity component.

There are many Bott towers $M_n(A)$ (e.g., a Hirzebruch surface $\mathcal{H}_a$ with $a \neq 0$) such that $\mathrm{Aut}_0(M_n(A))$ is not reductive. So by the Matushima–Lichnerowicz criterion [Lic58], many Bott manifolds do not admit Kähler metrics of constant scalar curvature. As pointed out by Mabuchi [Mab87], it follows from the work of Demazure [Dem70] that the reductivity of $\mathrm{Aut}_0(M_\Sigma)$ for a toric complex manifold M_Σ is equivalent to a balancing condition on a certain set $R(\Sigma)$ of *roots* associated to its fan Σ. A (Demazure) root of M_Σ is an element $\chi \in \mathfrak{t}^*$ which is dual to some normal $u_\rho \in \mathfrak{t}$ (for $\rho \in \Sigma_1$) in the sense that $\chi(u_\rho) = 1$ and $\chi(u_{\rho'}) \leq 0$ for all $\rho' \in \Sigma_1 \setminus \{\rho\}$. We then have the following result (see also the Demazure Structure Theorem in [Oda88, p. 140]).

Proposition 3.1 (Demazure)**.** *Let M_Σ be a smooth complete toric complex manifold with fan Σ.*

1. *The set $R(\Sigma) \subset \mathfrak{t}^*$ of roots of M_Σ is the root system of the algebraic group $\mathrm{Aut}_0(M_\Sigma)$ with respect to the maximal torus $\mathbb{T}^c$;*
2. *$\mathrm{Aut}_0(M_\Sigma)$ is reductive if and only if and only if $R(\Sigma) = -R(\Sigma)$.*

To apply this proposition to $M_n(A)$, let $\varepsilon_1, \ldots, \varepsilon_n$ be the basis of $\mathfrak{t}^*$ dual to $v_1, \ldots, v_n$.

Lemma 3.2. *We have $\varepsilon_i \in R(M_n(A))$ if and only if all entries of the ith row of A are non-negative. In particular, $\varepsilon_1 \in R(M_n(A))$. Similarly, $-\varepsilon_i \in R(M_n(A))$ if and only if all entries of the ith row of A (except $A_i^i = 1$) are non-positive, and $-\varepsilon_1 \in R(M_n(A))$.*

Proof. By definition $\varepsilon_i(v_j) = \delta_{ij}$, and

$$\varepsilon_i(u_j) = -\varepsilon_i\left(\sum_{k=j+1}^{n} A_i^j v_k\right) = -A_i^j$$

which is ≤ 0 for all j if and only if $A_i^j \geq 0$ for all j. Similarly $-\varepsilon_i(u_i) = A_i^i = 1$ and $-\varepsilon_i$ is ≤ 0 on all other normals if $A_i^j \leq 0$ for all $j \neq i$. □

In particular, ε_2 is a root if and only if $A_2^1 \geq 0$ and $-\varepsilon_2$ is a root if and only if $A_2^1 \leq 0$.

Corollary 3.3. *If for any row all the entries below the diagonal have the same sign and are not all zero, then* $\mathrm{Aut}_0(M_n(A))$ *is not reductive. In particular, if* $A_2^1 \neq 0$ *then* $\mathrm{Aut}_0(M_n(A))$ *is not reductive.*

This has a strong implication for Bott towers with topological twist zero.

Proposition 3.4. *Let* $M_n(A)$ *be a Bott tower with topological twist* 0 *and* $\mathrm{Aut}_0(M_n(A))$ *reductive. Then* $A = \mathbb{1}_n$ *and the Bott manifold is the product* $(\mathbb{CP}^1)^n$.

Proof. Suppose to the contrary that $\mathrm{Aut}_0(M_n(A))$ is reductive and $A \neq \mathbb{1}_n$. By Theorem 2.4, for such a Bott tower the A matrix takes the form $A = 2C_n \cdots C_1 - \mathbb{1}_n$ where C_k is a lower triangular unipotent matrix with at most one non-zero element $C_k^{i_k}$ below the diagonal and that lies in the kth row. Since $A \neq \mathbb{1}_n$ there exists $k \in \{2, \ldots, n\}$ such that $C_k \neq \mathbb{1}_k$ but $C_i = \mathbb{1}_i$ for $i < k$. So the kth row of C_k has the all 0 entries below the diagonal except in column $j < k$ whose entry is $C_k^j \neq 0$. But then from the form of A we see that the first row of A with non-zero entries below the diagonal has all zeroes except for $A_k^j = 2C_k^j \neq 0$. But this contradicts Corollary 3.3. □

Corollary 3.5. *Let* $M_n(A)$ *be a Bott tower with twist* t *and*

$$A = \begin{pmatrix} & \tilde{A} & & & \mathbb{0}_t & \\ A_{n-t+1}^1 & \cdots & A_{n-t+1}^{n-t} & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ A_n^1 & \cdots & A_n^{n-t} & A_n^{n-t+1} & \cdots & 1 \end{pmatrix}, \qquad A_j^i \in \mathbb{Z}, \tag{24}$$

where $\tilde{A} \neq \mathbb{1}_{n-t}$ *has topological twist* 0. *Then* $\mathrm{Aut}_0(M_n(A))$ *is not reductive.*

Example 3.1. We can obtain complete results in the case of a stage 3 Bott tower. Recall that the set Σ_1 of rays or 1-dimensional cones in Σ is

$$\Sigma_1 = \{\mathbb{R}_{\geq 0}v_1, \mathbb{R}_{\geq 0}v_2, \mathbb{R}_{\geq 0}v_3, \mathbb{R}_{\geq 0}(-v_3), \mathbb{R}_{\geq 0}(-v_2 - cv_3), \mathbb{R}_{\geq 0}(-v_1 - av_2 - bv_3)\}.$$

Applying Lemma 3.2 and Corollary 3.3, the reductive cases are as follows.

Corollary 3.6. *The connected component* $\mathrm{Aut}_0(M_3(a,b,c))$ *of the automorphism group of a stage* 3 *Bott tower* $M_3(a,b,c)$ *is reductive if and only if* $a = 0$ *and* $bc < 0$, *or* $a = b = c = 0$.

Thus, we see that the Bott tower $M_3(a,b,c)$ can admit a CSC Kähler metric only if $a = 0$ and either $bc < 0$ or $b = c = 0$.

3.2. **Divisors, support functions and primitive collections.** If M is a complex toric manifold (or orbifold) then any Weil divisor is Cartier; furthermore, any (integral, rational or real) divisor is linearly equivalent to a $\mathbb{T}^c$-invariant divisor, which in turn is a linear combination $\sum_{\rho\in\mathcal{S}} s_\rho D_\rho$ (with each s_ρ in $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$ respectively). Thus we may identify both the Picard group $\mathrm{Pic}(M)$ and the Chow group $A_{n-1}(M)$ with the quotient of $\mathbb{Z}^{\mathcal{S}} \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_{\mathcal{S}}, \mathbb{Z})$ by the inclusion $\mathbf{u}^\top\colon \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_{\mathcal{S}}, \mathbb{Z})$ sending λ to the principal ($\mathbb{Z}$-)divisor $\sum_{\rho\in\mathcal{S}} \lambda(u_\rho) D_\rho$. Similarly, we have an exact sequence,

$$0 \to \mathfrak{t}^* \to \mathbb{R}^*_{\mathcal{S}} \to H^2(M, \mathbb{R}) \to 0, \tag{25}$$

where we use $\mathfrak{t}^* = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{t}, \mathbb{R})$ and $H^2(M, \mathbb{R}) \cong \mathrm{Pic}(M) \otimes_{\mathbb{Z}} \mathbb{R}$. This last isomorphism identifies the Kähler cone $\mathcal{K}(M)$ in $H^2(M, \mathbb{R})$ with the ample cone $\mathcal{A}(M)$ in $\mathrm{Pic}(M) \otimes_{\mathbb{Z}} \mathbb{R} \cong A_{n-1}(M) \otimes_{\mathbb{Z}} \mathbb{R}$. We note also that the Picard number $\rho(M_n) = n$ for any Bott manifold M_n.

Let $PL(\Sigma)$ be the set of continuous real valued functions on $\mathfrak{t}$ which are linear on each cone in Σ. Since Σ is simplicial, the map $PL(\Sigma) \to \mathbb{R}^*_{\mathcal{S}} \cong \mathbb{R}_{\mathcal{S}}$ sending ψ to $(\psi(u_\rho))_{\rho\in\mathcal{S}}$ is a bijection. Hence any invariant divisor D has a unique *support function* $\psi_D \in PL(\Sigma)$ such that $D = \sum_{\rho\in\mathcal{S}} \psi_D(u_\rho) D_\rho$, and D is principal iff ψ_D is linear. Hence $H^2(M, \mathbb{R}) \cong PL(\Sigma)/\mathfrak{t}^*$. A function $\psi \in PL(\Sigma)$ is *convex*[2] if for all $x, y \in \mathfrak{t}$,

$$\psi(x) + \psi(y) \geq \psi(x + y);$$

then ψ is *strictly convex* if it is a different linear function on each maximal cone $\sigma \in \Sigma_n$. It is a classical result [Dan78] that $[D]$ is ample if and only if ψ_D is strictly convex. Then

$$P_D := \{\xi = \mathfrak{t}^* \mid \xi \leq \psi_D\}$$

is a convex polytope dual to Σ, i.e., its vertices correspond to maximal cones in Σ by assigning $\sigma \in \Sigma_n$ the linear form $\psi_D\big|_\sigma \in \mathfrak{t}^*$; P_D is often called a *Delzant polytope*.

As observed in [AGM93, Bat93, Cox97], the following notion of Batyrev [Bat91] (see also [CvR09, CLS11]) leads to a more explicit description of the Kähler cone $\mathcal{K}(M)$: we say $R \subseteq \mathcal{S}$, or the corresponding collection $\{u_\rho \mid \rho \in R\}$ of normals, is a *primitive collection* for a simplicial fan Σ, if R is a minimal element of $P(\mathcal{S}) \setminus \Phi$, i.e., $\{u_\rho \mid \rho \in R\}$ does not span a cone of Σ, but every proper subset does. Note that the primitive collections determine Φ, and that the open subset $\mathbb{C}_{\mathcal{S},\Phi}$ is the complement of the coordinate planes (closures of $\mathbb{C}_{\mathcal{S},R}$) corresponding to primitive collections R.

Theorem 3.7 ([Bat91, CvR09])**.** *A function $\psi \in PL(\Sigma)$ is convex if and only if for any primitive collection $R \subseteq \mathcal{S}$,*

$$\sum_{\rho\in R} \psi(u_\rho) \geq \psi\Big(\sum_{\rho\in R} u_\rho\Big).$$

It is strictly convex if this inequality is strict for all such R.

Since $M_n(A)$ is a quotient of $(\mathbb{C}^2_*)^n$, we see that the primitive collections in this case are the pairs $\{u_j, v_j\}$ for $j \in \{1, \ldots, n\}$. Hence the cones of the fan are the convex hulls of any subset of normals which does not contain any pair $\{u_j, v_j\}$. There are thus 2^n maximal cones, each containing precisely one normal from each pair $\{u_j, v_j\}$, $j \in \{1, \ldots, n\}$. Any such family of normals is a $\mathbb{Z}$-basis of Λ.

[2]Following [CvR09] the convention used here is opposite of the convention used in [CLS11].

3.3. **The ample cone and Kähler cone.** By applying Theorem 3.7 to a Bott tower $M_n(A)$, we obtain its ample cone $\mathcal{A}(M_n(A))$ and, by duality, its Kähler cone $\mathcal{K}(M_n(A))$. It will be of convenience later to consider polarized Bott towers $(M_n(A), [D])$ where D is an ample (integral) invariant Weil divisor. Since we are interested in the Kähler geometry of $M_n(A)$ we describe such polarized Bott towers in terms of the dual Kähler class $c_1([D])$.

Corollary 3.8. *An invariant $\mathbb{R}$-divisor D defines an ample class on $M_n(A)$ if and only if*

$$\psi_D(u_j) + \psi_D(v_j) > \psi_D(u_j + v_j) = \psi_D\Big(-\sum_{i=j+1}^{n} A_i^j v_i\Big) \tag{26}$$

for all $j \in \{1, \dots, n\}$.

If $\varepsilon_1, \dots, \varepsilon_n$ is the dual basis to $v_1, \dots, v_n$, then $\varepsilon_i(u_j) = -A_i^j$ and so the invariant principal divisors are spanned by

$$\sum_{\rho \in \mathcal{S}} \varepsilon_i(u_\rho) D_\rho = D_{v_i} - \sum_{j=1}^{n} A_i^j D_{u_j}. \tag{27}$$

Note that the divisors D_{v_i} and D_{u_i} are the zero and infinity sections of $\pi_i \colon \mathbb{P}(\mathbb{1} \oplus \mathcal{L}_i) \to M_{i-1}$ respectively. In particular, $\{D_{u_i}\}_i$ is a set of generators for the Chow group $A_{n-1}(M_n)$ of Weil divisors on M_n and the Poincaré dual of D_{u_i} can be identified with x_i. Hence the x_i's are algebraic classes.

We now write an arbitrary invariant divisor as

$$D = \sum_{\rho \in \mathcal{S}} s_\rho D_\rho = \sum_i s_i D_{u_i} + \sum_i t_i D_{v_i}. \tag{28}$$

From (27) we have the relations

$$D_{v_i} \sim D_{u_i} + \sum_{j=1}^{i-1} A_i^j D_{u_j}. \tag{29}$$

So writing $\psi_D(u_i) = s_i$ and $\psi_D(v_i) = t_i$ we note that the ampleness condition (26) becomes

$$s_j + t_j > \sum_{i=j+1}^{n} \psi_D(-A_i^j v_i) \tag{30}$$

which, since support functions are linear on the cones of a fan, becomes

$$s_j + t_j + \sum_{i=j+1}^{n} A_i^j t_i > 0 \tag{31}$$

when $A_i^j \leq 0$ for all $j < i$. Now using (29) we have

$$D \sim \sum_i s_i D_{u_i} + \sum_i t_i \Big(D_{u_i} + \sum_{j=1}^{i-1} A_i^j D_{u_j} \Big) = \sum_{j=1}^{n} \Big(s_j + t_j + \sum_{i=j+1}^{n} t_i A_i^j \Big) D_{u_j}. \tag{32}$$

Thus, D is ample by (31). So if we define $r_j = s_j + t_j + \sum_{i=j+1}^{n} t_j A_i^j$ we arrive at

Proposition 3.9. *If $A_i^j \leq 0$ the ample cone $\mathcal{A}(M_n(A))$, and thus the Kähler cone $\mathcal{K}(M_n(A))$, is the entire first orthant defined by $r_j > 0$ for all $j = 1, \ldots, n$.*

However, as we shall see, this is not always the case. An alternative approach is to consider all possible sets of generators of the Chow group $A_{n-1}(M_n(A))$. From (29) we see that there are precisely 2^{n-1} distinct sets of $\mathbb{T}^c$-invariant generators of $A_{n-1}(M_n(A))$. Thus, any ample $\mathbb{T}^c$-invariant divisor must take the form $D = r_1 D_{u_1} + \sum_{i=2}^n r_i D_i$ with $D_i = D_{u_i}$ or $D_i = D_{v_i}$. In principle the ample cone can be computed explicitly; however, we are only interested in certain special cases which we now discuss through a series of examples.

Example 3.2. We first consider the well-understood Hirzebruch surfaces $\mathcal{H}_a$. A Hirzebruch surface $\mathcal{H}_a = M_2(a)$, as a Bott tower with matrix A, $A_2^1 = a$, has normals $u_1 = -v_1 - av_2, u_2 = -v_2, v_1, v_2$. A diagram of the fan is given in [CLS11, Example 4.1.8]. [3] There are two primitive collections, $\{u_1, v_1\}$ and $\{u_2, v_2\}$, and by (27), the principal divisors are spanned by $D_{v_1} - D_{u_1}$ and $D_{v_2} - (aD_{u_1} + D_{u_2})$. Then an invariant divisor takes the form

$$D = \sum_{\rho \in \mathcal{S}} s_\rho D_\rho = r_1 D_{u_1} + r_2 D_2$$

where D_2 is either D_{u_2} or D_{v_2}. By (30) D defines an ample class if $r_2 > 0$ and $r_1 > \psi_D(-av_2)$. So if $a \leq 0$ we take $D_2 = D_{u_2}$ which gives $r_1 > 0$ and shows that in this case the ample cone $\mathcal{A}(\mathcal{H}_a)$, and hence the Kähler cone $\mathcal{K}(\mathcal{H}_a)$, is the entire first quadrant $r_1 > 0, r_2 > 0$.

If on the other hand $a > 0$ we take $D_2 = D_{v_2}$ giving $r_1 > \psi_D(-av_2) = \psi_D(au_2) = 0$ which again gives $\mathcal{A}(\mathcal{H}_a)$ as the entire first quadrant.

Next we consider the case of stage 3 Bott manifolds.

Example 3.3. We compute the ample cone for $n = 3$ in which case we have $a = A_2^1, b = A_3^1, c = A_3^2$. For a height 3 Bott tower $M_3(a, b, c)$, the fan $\Sigma_{a,b,c}$ has normals

$$u_1 = -v_1 - av_2 - bv_3, \quad u_2 = -v_2 - cv_3, \quad u_3 = -v_3, \quad v_1, \quad v_2, \quad v_3, \tag{33}$$

and by (27) the invariant principal divisors are spanned by

$$D_{v_1} - D_{u_1}, \quad D_{v_2} - (aD_{u_1} + D_{u_2}), \quad D_{v_3} - (bD_{u_1} + cD_{u_2} + D_{u_3}). \tag{34}$$

Writing $\psi_D(u_i) = s_i$ and $\psi_D(v_i) = t_i$ (for an invariant divisor D), we see by (26) that the ample cone (or equivalently the Kähler cone) is determined by the inequalities

$$s_1 + t_1 > \psi_D(-av_2 - bv_3), \quad s_2 + t_2 > \psi_D(-cv_3), \quad s_3 + t_3 > 0. \tag{35}$$

If $a, b, c \leq 0$, the first two relations reduce to $s_1 + t_1 + at_2 + bt_3 > 0$ and $s_2 + t_2 + ct_3 > 0$. Using (34) (with $s_{j'} = t_j$) we may write

$$D = \sum_{\rho \in \mathcal{S}} s_\rho D_\rho = \big(s_1 + t_1 + at_2 + bt_3\big) D_{u_1} + (s_2 + t_2 + ct_3) D_{u_2} + (s_3 + t_3) D_{u_3} \tag{36}$$

and so if we parametrize $H^2(M, \mathbb{R})$ by $r_1 PD([D_{u_1}]) + r_2 PD([D_{u_2}]) + r_3 PD([D_{u_3}]$, the Kähler cone is the first octant, $r_1 > 0, r_2 > 0, r_3 > 0$. Equivalently, $\{[D_{u_1}], [D_{u_2}], [D_{u_3}]\}$ is a set of generators for the Chow group $A_2(M_n)$ of Weil divisors on M_n.

[3] Note that our convention is the opposite to the convention used in [CLS11], that is our $\mathcal{H}_a$ is their $\mathcal{H}_{-a}$.

Now consider the general case where a, b, c have any signs. We consider the possible primitive collections whose elements are generators of $A_2(M_n)$. We note from the first of Equations (34) that $[D_{u_1}]$ and $[D_{v_1}]$ are equal, so there are only four such sets of generators, namely

$$\{[D_{u_1}], [D_{u_2}], [D_{u_3}]\}, \tag{37}$$

$$\{[D_{u_1}], [D_{u_2}], [D_{v_3}]\}, \tag{38}$$

$$\{[D_{u_1}], [D_{v_2}], [D_{u_3}]\}, \tag{39}$$

$$\{[D_{u_1}], [D_{v_2}], [D_{v_3}]\}. \tag{40}$$

Proposition 3.10. *For a height* 3 *Bott tower* $M_3(a, b, c)$ *there is a choice of invariant divisors* (37)–(40) *such that the ample cone is the full first octant if and only if one of the following two conditions hold*:

- $a \leq 0$ *and* $bc \geq 0$; *or*
- $a \geq 0$ *and* $(b - ac)c \geq 0$.

Proof. It is enough to check this on the four sets of generators given above, that is we write $D = r_1 D_1 + r_2 D_2 + r_3 D_3$ where $D_1 = D_{u_1}$, D_2 is either D_{u_2} or D_{v_2}, and D_3 is either D_{u_3} or D_{v_3}. Then (35) becomes

$$r_1 > \psi_D(-av_2 - bv_3), \quad r_2 > \psi_D(-cv_3), \quad r_3 > 0. \tag{41}$$

First suppose that $a \leq 0$; then in order that r_1 should give the full range $0 < r_1 < \infty$ the first of (41) implies that we should choose $D_2 = D_{u_2}$. In addition for both r_1 and r_2 to give the full range (41) implies that b and c cannot have opposite signs. If instead $b, c \leq 0$, then choosing $D = r_1 D_{u_1} + r_2 D_{u_2} + r_3 D_{u_3}$ shows that the ample cone is the entire first octant; whereas, if $b \geq 0,\ c > 0$ or $b > 0,\ c \geq 0$ we can choose $D = r_1 D_{u_1} + r_2 D_{u_2} + r_3 D_{v_3}$ to get the ample cone to be the entire first octant. This gives the first two conditions of the proposition. The case $a \geq 0$ is analogous, but also follows immediately from the equivalence $M_3(a, b, c) \sim M_3(-a, b - ac, c)$. □

Both of the conditions in Proposition 3.10 imply that $(2b - ac)c \geq 0$, i.e., $p_1(M)$ is a nonnegative multiple of $x_1 x_2 = (1/2)c_1(O(1, 1))^2$.

Remark 3.11. Here we enumerate the complimentary inequalities to those of Proposition 3.10, that is, those cases where the ample cone is not the full first octant. From the proposition if $a \leq 0$ we obtain the full first octant unless, $b < 0$ and $c > 0$, or $b > 0$ and $c < 0$. Since $a \leq 0$ we write $D = r_1 D_{u_1} + r_2 D_{u_2} + r_3 D_3$, and since $c < 0$ if we choose $D_3 = D_{v_3}$ we get $r_2 > \psi_D(-cv_3) = -cr_3$; whereas, if we choose $D_3 = D_{u_3}$ we get $r_1 > br_3$. In either case we fail to get the full first octant. On the other hand if $a > 0$ we do not obtain the entire first octant if $c \leq 0$ and $b > ac$, or if $c \geq 0$ and $b < ac$. If we choose $D = r_1 D_{u_1} + r_2 D_{v_2} + r_3 D_3$ with $D_3 = D_{u_3}$ we get when $c \leq 0$

$$r_1 > \psi_D(-av_2 - bv_3) = \psi_D(a(u_2 + cv_3) - bv_3) = \psi_D(au_2 - (b - ac)v_3) = \psi_D(au_2 + (b - ac)u_3)$$

which implies $r_1 > (b - ac)r_3, r_2 > 0$ when $b > ac$; whereas, if we choose $D_3 = D_{v_3}$ this and the second of Equations (35) gives $r_1 > 0, r_2 > -cr_3$. A similar analysis holds for the remaining case.

Example 3.4. Here we consider again the Bott towers $M_{N+1}(\mathbf{k})$ of Example 1.1 whose A matrix takes the form (5). For these Bott manifolds we write $D = \sum_{i=1}^{N} r_i D_{u_i} + r_{N+1} D_{N+1}$. Then from (30) the ample cone is determined by inequalities

$$r_j > \psi_D(-k_j v_{N+1}) = \psi_D(k_j u_{N+1}) \text{ and } r_{N+1} > 0.$$

Hence, if $k_j \leq 0$ for all $j = 1, \ldots, N$ we choose $D_{N+1} = D_{u_{N+1}}$ giving $\mathcal{A}(M_{N+1}(A))$ to be the entire first orthant; whereas, if $k_j \geq 0$ for all $j = 1, \ldots, N$ we choose $D_{N+1} = D_{v_{N+1}}$ again giving $\mathcal{A}(M_{N+1}(\mathbf{k}))$ to be the entire first orthant. However, when not all k_j have the same sign, we do not get the entire first orthant. Without loss of generality we assume that $k_j \leq 0$ for $j = 1, \ldots, m$ and $k_j > 0$ for $j = m+1, \ldots, N$. If $m \leq \lfloor \frac{N}{2} \rfloor$ we choose $D_{N+1} = D_{v_{N+1}}$, and if $m > \lfloor \frac{N}{2} \rfloor$ we choose $D_{N+1} = D_{u_{N+1}}$. In the former case the ample cone is given by $r_j > -k_j r_{N+1}$ for $j = 1, \ldots, m$ and $r_j > 0$ for $j = m+1, \ldots, N+1$; whereas, in the latter case it is determined by $r_j > 0$ for $j = 1, \ldots, m, N+1$ and $r_j > k_j r_{N+1}$ for $j = m+1, \ldots, N$.

Example 3.5. For the Bott towers $M_{N+1}(\mathbf{l}, \mathbf{k})$ of Example 1.2, the primitive collections are $\{u_j, v_j\} : j \in \{1, \ldots, N+1\}$ where $u_j = -v_j - l_j v_N - k_j v_{N+1}$ for $j = 1, \ldots, N-1$, $u_N = -v_N - k_N v_{N+1}$, and $u_{N+1} = -v_{N+1}$. So an invariant divisor can be written as

$$D = \sum_{i=1}^{N-1} r_i D_{u_i} + r_N D_N + r_{N+1} D_{N+1}$$

where D_N is D_{v_N} or D_{u_N} and D_{N+1} is $D_{v_{N+1}}$ or $D_{u_{N+1}}$. So from (30) we have $r_{N+1} > 0$, and $r_N > \psi_D(-k_N v_{N+1}) = \psi_D(k_N u_{N+1})$ and for $j = 1, \ldots, N-1$,

$$\begin{aligned} r_j > \max\{&\psi_D(-l_j v_N - k_j v_{N+1}), \psi_D(-l_j v_N + k_j u_{N+1}), \\ &\psi_D(+l_j u_N - k_j v_{N+1}), \psi_D(+l_j u_N + k_j u_{N+1})\}. \end{aligned} \tag{42}$$

One can proceed with an analysis similar to Example 3.3 to determine for which $M_{N+1}(\mathbf{l}, \mathbf{k})$ the Kähler cone is the entire first orthant, and for which it is not; however, here we just consider a special case of interest in Proposition 5.5 below.

Example 3.5.1. We take $N = 2m+1$ with the matrix

$$A = \begin{pmatrix} \mathbb{1}_m & \mathbb{0}_m & \mathbf{0}_m & \mathbf{0}_m \\ \mathbb{0}_m & \mathbb{1}_m & \mathbf{0}_m & \mathbf{0}_m \\ \vdots & \vdots & \vdots & \vdots \\ q\mathbf{1}_m & -q\mathbf{1}_m & 1 & 0 \\ q(k+1)\mathbf{1}_m & q(k-1)\mathbf{1}_m & 2 & 1 \end{pmatrix} \tag{43}$$

with $q, k-1 \in \mathbb{Z}^+$, and where $\mathbb{1}_m$ is the m by m identity matrix, $\mathbb{0}_m$ the m by m zero matrix, $\mathbf{0}_m$ is the zero column vector in $\mathbb{R}^m$, and $\mathbf{1}_m = (1\,1 \ldots 1)$ is a row vector in $\mathbb{R}^m$.

Proposition 3.12. *The the ample cone, hence the Kähler cone, of the Bott tower $M_{2m+2}(A)$ with A given by (43) is represented by*

$$r_j > 0 \text{ for } j = 1, \ldots, m; \quad r_j > q r_{2m+1} \text{ for } j = m+1, \ldots, 2m;$$
$$r_{2m+1} > 0; \quad r_{2m+2} > 0.$$

Proof. To determine the ample cone, we note that $r_{2m+2} > 0$, $r_{2m+1} > \psi_D(-2v_{2m+2}) = \psi_D(2u_{2m+2})$, and from (42) we get for $j = 1, \dots, m$ that $r_j > \psi_D(qu_{2m+1} + q(k-1)u_{2m+2}$. So we choose $D_{2m+1} = D_{v_{2m+1}}$ and $D_{2m+2} = D_{v_{2m+2}}$. But then for $j = m+1, \dots, 2m$ we get $r_j > \psi_D(qv_{2m+1} - q(k-1)v_{2m+2}) = qr_{2m+1}$ which proves the result. □

Since the last row of the matrix (43) has all positive entries, the connected component of the automorphism group of $M_{2m+2}(A)$ is not reductive by Corollary 3.2. It follows that $M_{2m+2}(A)$ does not admit a CSC Kähler metric.

3.4. **Fano Bott manifolds.** It follows immediately from the fact that Bott manifolds are compact toric or from Equations (3), (11), and (14) that the first Chern class of a Bott manifold is either positive definite or indefinite, and for most it is indefinite. Recall that a complex manifold (M, J) is *Fano* if the first Chern class $c_1(M)$ is positive definite, that is, $c_1(M)$ lies in the Kähler cone, in which case, the Kähler classes $[\omega]$ in the span of $c_1(M)$ (and any Kähler metrics ω in such a class) are said to be *monotone*. Equivalently, the anti-canonical divisor lies in the ample cone. There is a lot known about toric Fano manifolds with $n \leq 4$ (see for example [Bat81, WW82, Sak86, Mab87, Bat91, Nak93, Nak94, Bat99, BS99]).

Fano Bott manifolds with topological twist 0.

Proposition 3.13. *The only Fano Bott tower $M_n(A)$ with topological twist* 0 *is the product $(\mathbb{CP}^1)^n$ represented by $A = \mathbb{1}_n$.*

Proof. If $M_n(A)$ is a Bott tower with topological twist 0, then by Theorem 2.4 A must satisfy $A = 2C_n \cdots C_1 - \mathbb{1}_n$ where C_1 is the identity matrix and for each k the unipotent matrix C_k can have at most one non-zero element in its kth row below the diagonal. Hence, any non-zero off-diagonal element of A must be a multiple of 2, alternatively,

$$A = \mathbb{1}_n + 2L \tag{44}$$

where $L = C_n \cdots C_1 - \mathbb{1}_n$ is an n step nilpotent matrix, i.e., $L^n = 0$. By Lemma 1.10 $M_n(A^{-1})$ is equivalent to $M_n(A)$ so it has topological twist 0 and as in (15), A^{-1} can be represented as

$$A^{-1} = 2C'_n \cdots C'_1 - \mathbb{1}_n = \mathbb{1}_n + 2L' \tag{45}$$

where C'_k has the same property as C_k. From Equations (15), (44), and (45) we obtain

$$AL' = -L. \tag{46}$$

Now consider the first Chern class. Using vector notation we have

$$c_1(M_n) = \sum_{i=1}^{n}(x_i + y_i) = \Sigma(\mathbb{1}_n + A)\mathbf{x} = 2\Sigma C_n \cdots C_1\mathbf{x} = 2\Sigma(\mathbb{1}_n + L)\mathbf{x}$$

where Σ denotes the sum of the components. This implies that the non-zero elements of L must be positive which implies that the off-diagonal elements of A are non-negative. But we can also give $c_1(M_n)$ in terms of the basis $\{y_i\}$ as well

$$c_1(M_n) = \sum_{i=1}^{n}(x_i + y_i) = \Sigma(\mathbb{1}_n + A^{-1})\mathbf{y} = 2\Sigma(\mathbb{1}_n + L')\mathbf{y}$$

So as above the non-zero elements of L' must also be positive. But this violates Equation (46) unless the off-diagonal elements of A vanish. □

Fano Bott manifolds with topological twist 1. For the Bott towers $M_{N+1}(\mathbf{k})$, the *monotone* case when $c_1(M_{N+1}(\mathbf{k}))$ is a Kähler class which has been well studied [KS86, Koi90].

Proposition 3.14. *A Fano Bott tower of the type $M_{N+1}(\mathbf{k})$ can be put in the form*

$$M_{N+1}(\mathbf{k}) \cong M_m((\overbrace{\pm 1, \dots, \pm 1}^{m-1})) \times \overbrace{\mathbb{CP}^1 \times \cdots \times \mathbb{CP}^1}^{N+1-m}$$

for some $m = 2, \dots, N+1$. Moreover, $M_{N+1}(\mathbf{k})$ with the monotone Kähler class admits a Kähler–Ricci soliton which is Kähler–Einstein if and only if the number of -1 equals the number of $+1$ in $\mathbf{k}$.

Proof. For the Bott towers $M_{N+1}(\mathbf{k})$ we have

$$c_1(M_{N+1}(\mathbf{k})) = \sum_{i=1}^{N}(2+k_i)x_i + 2x_{N+1} = \sum_{i=1}^{N}(2-k_i)x_i + 2y_{N+1}.$$

So $M_{N+1}(\mathbf{k})$ is Fano if and only if $k_i = 0, \pm 1$. By an element of BC_n we can put $\mathbf{k}$ in the form $(1, \dots, 1, -1, \dots, -1, 0 \dots, 0)$ where the number of 0's is $N+1-m$, the number of $+1$'s is r and the number of -1's is s with $r+s = m-1$ which implies the given form.

It follows from [ACGT08b, Section 3] together with [Koi90, WZ04] that $M_{N+1}(\mathbf{k})$ admits a Kähler–Ricci soliton which is Kähler–Einstein if and only if the number of -1's in $\mathbf{k}$ equals the number of $+1$'s in $\mathbf{k}$. □

Fano Bott manifolds with topological twist 2. For the Bott towers $M_{N+1}(\mathbf{l}, \mathbf{k})$ we have

$$\begin{aligned} c_1(M_{N+1}(\mathbf{l}, \mathbf{k})) &= \sum_{i=1}^{N-1}(2+l_i+k_i)x_i + (2+k_N)x_N + 2x_{N+1} \\ &= \sum_{i=1}^{N-1}(2+l_i-k_i)x_i + (2-k_N)x_N + 2y_{N+1} \\ &= \sum_{i=1}^{N-1}(2-l_i+k_i-k_Nl_i)x_i + (2+k_N)y_N + 2x_{N+1} \\ &= \sum_{i=1}^{N-1}(2-l_i-k_i+k_Nl_i)x_i + (2-k_N)y_N + 2y_{N+1} \end{aligned} \tag{47}$$

Lemma 3.15. *If $M_{N+1}(\mathbf{l}, \mathbf{k})$ is Fano then $l_i = 0, \pm 1$ and $k_i = 0, \pm 1$.*

Proof. That $k_N = 0, \pm 1$ is obvious, and that $l_i = 0, \pm 1$ follows from adding the first two and the last two equations. Then one checks that $k_i = 0, \pm 1$ as well. □

However, not all possibilities in Lemma 3.15 can occur. For example, from the first two of Equations (47) we see that for $i = 1, \dots, N-1$ if $l_i = -1$ then we must have $k_i = 0$; whereas, if $l_i = 1$ then we must have $k_i = k_N = -1$ or $k_i = k_N = 1$. Referring to the equivalence relations of Example 1.5 we see that this last case implies that

$k_i' = k_i - k_N = 0$. So up to fiber inversion equivalence we can assume that $l_i = -1$ or 0. If $l_i = 0$ then all possibilities for k_i, k_N can occur. Since we have already considered Bott towers of twist ≤ 1, we can assume that not all l_i nor all k_i vanish. Up to equivalence we can assume that $l_i = -1$ and $k_i = 0$ for $i = 1, \ldots, m$, and $l_i = 0$ for $i = m+1, \ldots, N-1$ with $k_i = 0, \pm 1$ for $i = m+1, \ldots, N$ and not all k_i vanish. So we have arrived at

Proposition 3.16. *The Bott tower $M_{N+1}(\mathbf{l}, \mathbf{k})$ is Fano if and only if, up to equivalence, the matrix A takes the form*

$$A = \begin{pmatrix} \mathbb{1}_r & \mathbb{0}_{r\times s} & 0 & \cdots & 0 & 0 \\ \mathbb{0}_{s\times r} & \mathbb{1}_s & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\mathbf{1}_r & \mathbf{1}_s & 0 & \cdots & 1 & 0 \\ \mathbf{0}_r & k_N\mathbf{1}_s & k_m & \cdots & k_N & 1 \end{pmatrix}, \tag{48}$$

where $\mathbb{1}_r$ is the r by r identity matrix, $r + s = m - 1$, $\mathbf{1}_r = (\overbrace{1, \ldots, 1}^{r})$, and for $i = m, \ldots, N$, $k_i = 0, \pm 1$ and not all k_i vanish.

These Bott manifolds admit Kähler–Ricci solitons by [WZ04] and a Kähler–Einstein metric when the Futaki invariant vanishes. It would be interesting to see if there are any new Kähler–Einstein metrics in this class of Bott manifolds. It follows from the Nakagawa's classification [Nak93, Nak94] of Kähler–Einstein Fano toric 4-folds that there are no such Kähler–Einstein metrics when $N = 3$. In fact, from the classification we have

Corollary 3.17. *The only stage 4 Bott manifolds that admit Kähler–Einstein metrics are the standard product $(\mathbb{CP}^1)^4$ and $\mathbb{P}(\mathbb{1} \oplus \mathcal{O}(1, -1)) \times \mathbb{CP}^1$.*

We mention also that the Bott towers given by the matrix A in (43) are not Fano.

Example 3.6. We specialize these results to the height 3 Bott towers $M_3(a, b, c)$. Here, the first Chern class in the distinguished bases is

$$\begin{aligned} c_1(M_3) &= (2 + a + b)x_1 + (2 + c)x_2 + 2x_3, \\ &= (2 + a - b)x_1 + (2 - c)x_2 + 2y_3, \\ &= (2 - a + b - ac)x_1 + (2 + c)y_2 + 2x_3, \\ &= (2 - a - b + ac)x_1 + (2 - c)x_2 + 2x_3. \end{aligned}$$

Proposition 3.18. *A height 3 Bott tower $M_3(a, b, c)$ is Fano if and only if (a, b, c) is one of the following:*

- $(1, 0, 0), ((1, 1, 1), (1, -1, -1),$
- $(0, b, c)$ *with* $b = 0, \pm 1$ *and* $c = 0, \pm 1$.
- $(-1, 0, 0), (-1, 0, 1), (-1, 0, -1)$.

The Bott manifolds are the equivalences classes described in Example 1.6. We give a representative in each equivalence class. Explicitly,

Proposition 3.19. *Up to equivalence a Fano stage 3 Bott manifold is one of the following*

$$M_3, (0, 0, 0),\ M_3(0, 0, 1),\ M_3(0, 1, 1),\ M_3(0, 1, -1),\ M_3(-1, 0, 1).$$

The first Pontrjagin class $p_1(M_3(a,b,c)) = c(2b-ac)x_1x_2$ is a diffeomorphism invariant. We have

$$p_1(M_3(0,0,0)) = p_1(M_3(0,0,1)) = 0, \qquad p_1(M_3(0,1,1)) = 2x_1x_2,,$$
$$p_1(M_3(0,1,-1)) = -2x_1x_2, \qquad p_1(M_3(-1,0,1)) = x_1x_2.$$

The first of these manifolds is the $(\mathbb{CP}^1)^3$, and the second is $\mathbb{CP}^1 \times \mathcal{H}_1$ which has topological twist 1. The third and fourth are $\mathbb{CP}^1$ bundles over $\mathbb{CP}^1 \times \mathbb{CP}^1$ which also have topological twist 1, while the last is a Bott manifold with topological twist 2. Note that the Picard number of a stage 3 Bott manifold is 3, so it must be of type III in the list of the main theorem in [WW82]. By [WZ04] the monotone Kähler class admits a Kähler–Ricci soliton; however, from [Mab87, Remark 2.5] we have

Corollary 3.20. *The only stage* 3 *Bott manifolds that admit a Kähler–Einstein metric are the standard product* $M_3(0,0,0) = (\mathbb{CP}^1)^3$ *and the Koiso–Sakane projective bundle* $M_3(0,1,-1) = \mathbb{P}(\mathbb{1} \oplus \mathcal{O}(1,-1))$.

4. The symplectic viewpoint

4.1. **Toric symplectic manifolds and finiteness.** Any Kähler manifold (M, g, J, ω) has an underlying symplectic manifold (M, ω). The symplectic viewpoint studies conversely the space of complex structures J on a given symplectic manifold (M, ω) for which ω is the Kähler form of a Kähler metric g on (M, J).

Definition 4.1. A complex n-manifold M is *compatible* with a symplectic $2n$-manifold (X, ω) if there is a diffeomorphism $f\colon M \to X$ such that $f^*\omega$ is a Kähler form on M. We also say that the symplectic manifold (X, ω) is *of Kähler type* (with respect to M).

If $f\colon M \to X$ is such a diffeomorphism and M is toric with respect to a complex n-torus $\mathbb{T}^c$, then $T^n = f_*(\mathbb{T}^c) \cap \mathrm{Symp}(X, \omega)$ is a (real) hamiltonian n-torus in $\mathrm{Symp}(X, \omega)$ with momentum map $\mu\colon X \to \mathfrak{t}^*$ (where $\mathfrak{t}$ is the Lie algebra of T^n). Then (X, ω, T^n) is a *toric symplectic manifold* and (for X compact) the image of μ is a compact convex polytope P in $\mathfrak{t}^*$, called the *Delzant polytope* of (X, ω, T^n), such that the collection of cones in $\mathfrak{t}$ dual to the faces of P is the fan of M.

Hamiltonian n-tori T^n and $\tilde{T}^n$ in $\mathrm{Symp}(X, \omega)$ define *equivalent* toric symplectic manifolds if there is a symplectomorphism that intertwines them, i.e., they are conjugate as subgroups of $\mathrm{Symp}(X, \omega)$. Thus the set $\mathcal{C}_n(X, \omega)$ of conjugacy classes of hamiltonian n-tori in $\mathrm{Symp}(X, \omega)$ parametrizes equivalence classes of toric symplectic structures on (X, ω). In [McD11, Prop. 3.1], McDuff shows that the set $\mathcal{C}_n(X, \omega)$ is finite.

Proposition 4.2. *Let* (X, ω) *be a symplectic* $2n$*-manifold. Then there are finitely many biholomorphism classes of toric complex manifolds compatible with* (X, ω)*, and this set is naturally a quotient of* $\mathcal{C}_n(X, \omega)$.

Proof. A hamiltonian n-torus T^n in $\mathrm{Symp}(X, \omega)$ determines a Delzant polytope P in $\mathfrak{t}^*$ and hence a fan in $\mathfrak{t}$. Let M be the toric complex manifold constructed from the fan as a quotient of $\mathbb{C}_{\mathcal{S}}$ where $\mathcal{S}$ is the set of rays in the fan. Then Delzant's Theorem [Del88] says that (X, ω) is T^n-equivariantly symplectomorphic to a symplectic quotient of $\mathbb{C}_{\mathcal{S}}$ canonically diffeomorphic to M, in such a way that M is compatible with (X, ω) and T^n is the induced torus. Furthermore, if T^n and $\tilde{T}^n$ are conjugate hamiltonian n-tori, then

by [Del88], their Delzant polytopes P and $\tilde{P}$ are equivalent by an affine transformation $\mathfrak{t} \to \tilde{\mathfrak{t}}$ whose linear part is integral, hence the associated fans are equivalent, and so the corresponding toric complex manifolds are equivariantly biholomorphic—see for example [Oda88, Theorem 1.13].

Thus there is a well-defined map from $\mathcal{C}_n(X,\omega)$ to biholomorphism classes of toric complex manifolds compatible with (X,ω), and any such biholomorphism class arises in this way, so the map is a surjection. The biholomorphism classes thus form a quotient of $\mathcal{C}_n(X,\omega)$, which is finite by [McD11, Prop. 3.1]. □

To apply this result to Bott towers, we let $\mathcal{BT}^n(X,\omega)$ denote the set of Bott towers that are compatible with (X,ω) and $\mathcal{B}_n(X,\omega)$ the set of their biholomorphism classes (elements of $\mathcal{B}_n = \mathcal{BT}_0^n/\mathcal{BT}_1^n$). When this set is nonempty, we say (X,ω) has *Bott type*. If $M_n(A)$ is compatible with (X,ω), so are all elements of its biholomorphism class.

Theorem 4.3. *Let (X,ω) be a symplectic $2n$-manifold of Bott type. Then there is a canonical surjection $\mathcal{C}_n(X,\omega) \to \mathcal{B}_n(X,\omega)$ sending the conjugacy class of a hamiltonian n-torus T^n to the class of Bott towers compatible with (X,ω) and T^n. In particular, $\mathcal{B}_n(X,\omega)$ is finite.*

Proof. Since (X,ω) is of Bott type, its cohomology ring is a Bott quadratic algebra of rank n by [CS11, Theorem 4.2]. Hence if M is a toric complex manifold compatible with (X,ω), the Delzant polytope of the induced toric symplectic structure is combinatorially equivalent to an n-cube. But then [MP08, Theorem 3.4] implies that M is equivariantly biholomorphic to a Bott tower $M_n(A)$. The result now follows from Proposition 4.2. □

It is natural to ask whether the map $\mathcal{C}_n(X,\omega) \to \mathcal{B}_n(X,\omega)$ is a bijection. For this, suppose T^n and $\tilde{T}^n$ have the same image. Then there is a diffeomorphism f of (X,ω) intertwining T^n and $\tilde{T}^n$. Hence (X,ω,T^n) and $(X,f^*\omega,T^n)$ are toric symplectic manifolds which are symplectomorphic (by f).

4.2. **Symplectic products of spheres.** We are interested in finding the compatible Bott structures to a given symplectic manifold for certain cases. We need to fix the diffeomorphism type and then the symplectic form.

The product of 2-spheres $(S^2)^n = (\mathbb{CP}^1)^n$ admits symplectic forms $\omega_{\mathbf{k}} = \sum_{i=1}^n k_i\omega_i$ where $k_i \in \mathbb{R}^+$ and ω_i is the standard volume form on the ith factor of $(S^2)^n$. In this case we know from Theorem 2.4 that this diffeomorphism type is determined entirely by its integral graded cohomology ring.

Lemma 4.4. *Let $\omega_{\mathbf{k}} = \sum_{i=1}^n k_i\omega_i$ and $\omega_{\mathbf{k}'} = \sum_{i=1}^n k_i'\omega_i$ be symplectic forms on $(S^2)^n$ with $k_i, k_i' \in \mathbb{R}^+$, and suppose $((S^2)^n,\omega_{\mathbf{k}})$ and $((S^2)^n,\omega_{\mathbf{k}'})$ are symplectomorphic. Then they are equivalent as toric symplectic manifolds.*

Proof. Let $f\colon (S^2)^n \to (S^2)^n$ be a diffeomorphism such that $f^*\omega_{\mathbf{k}} = \omega_{\mathbf{k}'}$. This induces a linear map f_* on $H^2((S^2)^n,\mathbb{Z}) \cong \mathbb{Z}^n$ via the basis $x_1,\dots x_n$ of standard area forms on the S^2 factors. Now f_* must preserve the cup product relations $x_i \cup x_i = 0$. This implies that the matrix B of f_* with respect to $x_1,\dots x_n$ satisfies the relations $B_i^j B_i^k = 0$ for all i and $j < k$. Since $\det B = \pm 1$, B is a monomial matrix whose nonzero entries are ± 1. Hence $k_i = \pm k'_{\sigma(i)}$ for some permutation $\sigma \in S_n$, and the signs must be positive since $k_i, k_i' \in \mathbb{R}^+$. It follows that $((S^2)^n,\omega_{\mathbf{k}})$ and $((S^2)^n,\omega_{\mathbf{k}'})$ are equivalent by permuting the factors of $(S^2)^n$. □

Proposition 4.5. *Let the Bott tower $M_n(A)$ be diffeomorphic to $(S^2)^n$ with a split symplectic form $\omega_{\mathbf{k}} = \sum_{i=1}^n k_i\omega_i$ where $k_i \in \mathbb{R}^+$ and ω_i is the standard volume form on the ith factor of $(S^2)^n$. Suppose further that $A_i^j \leq 0$ for all $j < i$. Then with an appropriate choice of order $[\omega]$ is a Kähler class if and only if $\sum_{i=1}^n k_i(\delta_i^j - m_i^j) > 0$ and $\alpha_j^2 = 0$ for each $j = 1, \ldots, n$ where $m_i^j = -\frac{1}{2}A_i^j$ if $j < i$ and $m_i^j = 0$ if $j \geq i$.*

Proof. Up to order and sign $[\omega_1], \ldots, [\omega_n]$ is the unique basis of square-zero generators of the cohomology ring $H^*(M_n(A), \mathbb{Z})$. Thus, if $M_n(A)$ is diffeomorphic to $(S^2)^n$, by Lemma 2.5, we can identify $[\omega_i]$ with

$$x_i + \frac{\alpha_i}{2} = x_i + \frac{1}{2}\sum_{j=1}^{i-1} A_i^j x_j. \tag{49}$$

In particular, the entries A_i^j with $j < i$ must be even. Let us consider the matrix whose entries satisfy $m_i^j = -\frac{1}{2}A_i^j$ for all $j < i$ and zero elsewhere. So the square-zero classes are $x_i - \sum_{j=1}^{i-1} m_i^j x_j$ with $m_i^j \geq 0$. Thus, using Equation (49) we have

$$[\omega] = \sum_i k_i[\omega_i] = \sum_i k_i(x_i - \sum_{j=1}^n m_i^j x_j) = \sum_{i=1}^n \sum_{j=1}^n k_i(\delta_i^j - m_i^j)x_j.$$

Then since x_i is the Poincaré dual $PD(D_{u_i})$, Proposition 3.9 implies that $[\omega]$ is a Kähler class if and only if $\sum_{i=1}^n k_i(\delta_i^j - m_i^j) > 0$ for each j.

Furthermore, since M_n is diffeomorphic to $(S^2)^n$ we know that the total Pontrjagin class is trivial, which by Equation (12) is equivalent to $\alpha_j^2 = 0$ for all j. □

The compatibility problem is complicated for arbitrary n, so we restrict ourselves to $n = 2, 3$. First we take $n = 2$ with diffeomorphism type $S^2 \times S^2$, and take the split symplectic structure as

$$\omega_{k_1,k_2} = k_1\omega_1 + k_2\omega_2$$

where ω_i is the pullback of the volume form by the projection map $\mathrm{pr}_i \colon S^2 \times S^2 \to S^2$ onto the ith factor and $k_1, k_2 \in \mathbb{R}^+$. Now we know that it is precisely the even Hirzebruch surfaces that are diffeomorphic to $S^2 \times S^2$. Let us take $a \leq 0$ and even, so we set $a = -2m$ with $m \in \mathbb{Z}_{\geq 0}$. Since x_i is the Poincaré dual of $[D_{u_i}]$, from (49) we can write

$$[\omega_{k_1,k_2}] = k_1x_1 + k_2(x_2 - mx_1) = (k_1 - mk_2)x_1 + k_2x_2 = (k_1 - mk_2)\mathrm{PD}(D_{u_1}) + k_2\mathrm{PD}(D_{u_2}).$$

Thus, the symplectic manifold $(S^2 \times S^2, \omega_{k_1,k_2})$ is of Bott type with respect to $\mathcal{H}_a = \mathcal{H}_{-2m}$ if and only if $m < \frac{k_1}{k_2}$. So the number of compatible complex structures is $\lceil \frac{k_1}{k_2} \rceil$. Karshon proves in [Kar03] that $\lceil \frac{k_1}{k_2} \rceil$ is the number of conjugacy classes of maximal tori in the symplectomorphism group.

In the case that a is positive, we put $a = 2m$ and use $PD(D_{u_1})$ and $PD(D_{v_2})$ as the algebraic generators stipulated above. We have using $\mathrm{PD}(D_{v_2}) = x_2 - 2mx_1$ that

$$[\omega_{k_1,k_2}] = k_1x_1 + k_2(x_2 + mx_1)) = (k_1 - mk_2)\mathrm{PD}(D_{u_1}) + k_2\mathrm{PD}(D_{v_2}).$$

So for m positive we have $k_1 > 0, k_2 > mk_1$. So the symplectic manifold $(S^2 \times S^2, \omega_{k_1,k_2})$ is of Bott type with respect to $\mathcal{H}_{2m}$ for any $m \in \mathbb{Z}$ if and only if $k_2 > 0, k_1 > |m|k_2$. It is known that the cases m and $-m$ are equivalent as toric complex manifolds, but not as Bott towers.

Next we consider the case $n = 3$. For the symplectic viewpoint we first fix a diffeomorphism type. From Proposition 2.9 we know that $M_3(a,b,c)$ is diffeomorphic to $(S^2)^3$ if and only if $c(2b-ac) = 0$, and a, b and c are all even. We now assume the latter by replacing (a,b,c) by $(2a,2b,2c)$ and consider the Bott tower $M_3(2a,2b,2c)$ which is diffeomorphic to $(S^2)^3$ with the symplectic form

$$\omega_{k_1,k_2,k_3} = k_1\omega_1 + k_2\omega_2 + k_3\omega_3 > 0 \tag{50}$$

where ω_i is the standard area form on the ith factor S^2 and $k_i \in \mathbb{R}^+$, and as before the k_i are ordered $0 < k_3 \leq k_2 \leq k_1$.

Theorem 4.6. *The symplectic manifold $((S^2)^3, \omega_{k_1,k_2,k_3})$ is of Bott type with respect to $M_3(2a,2b,2c)$ if and only if one of the following hold*:

(1) $c = 0$ *with* $k_1 - |a|k_2 - |b|k_3 > 0,\ k_2 > 0,\ k_3 > 0$.
(2) $c \neq 0$ *and* $b = ac$ *with* $k_1 - |a|(k_2 - |c|k_3) > 0,\ k_2 - |c|k_3 > 0,\ k_3 > 0$.

Proof. First applied to $M_3(2a,2b,2c)$ the $p_1(M) = 0$ constraint now takes the form $c(b-ac) = 0$. So we have the two cases enunciated in the theorem which reduce to the Bott towers $M_3(2a,2b,0)$ and $M_3(2a,2ac,2c)$, respectively. Now using Equations (49) and (11) we can rewrite the symplectic class (50) in terms of the 4 sets of preferred bases in Σ_3 giving

$$[\omega_{k_1,k_2,k_3}] = (k_1 + ak_2 + bk_3)x_1 + (k_2 + ck_3)x_2 + k_3x_3, \tag{51}$$

$$[\omega_{k_1,k_2,k_3}] = (k_1 + ak_2 - bk_3)x_1 + (k_2 - ck_3)x_2 + k_3y_3, \tag{52}$$

$$[\omega_{k_1,k_2,k_3}] = (k_1 - ak_2 + (b-2ac)k_3)x_1 + (k_2 + ck_3)y_2 + k_3x_3, \tag{53}$$

$$[\omega_{k_1,k_2,k_3}] = (k_1 - ak_2 - (b-2ac)k_3)x_1 + (k_2 - ck_3)y_2 + k_3y_3. \tag{54}$$

To assure that $[\omega_{k_1,k_2,k_3}]$ is a Kähler class the coefficients must be positive in each basis. When $c = 0$ the positivity of the coefficients in these equations is equivalent to $k_1 - |a|k_2 - |b|k_3 > 0,\ k_2 > 0,\ k_3 > 0$. However, when $c \neq 0$ and $b = ac$ the equations reduce to

$$\begin{aligned}
[\omega_{k_1,k_2,k_3}] &= (k_1 + ak_2 + ack_3)x_1 + (k_2 + ck_3)x_2 + k_3x_3,\\
[\omega_{k_1,k_2,k_3}] &= (k_1 + ak_2 - ack_3)x_1 + (k_2 - ck_3)x_2 + k_3y_3,\\
[\omega_{k_1,k_2,k_3}] &= (k_1 - ak_2 - ack_3)x_1 + (k_2 + ck_3)y_2 + k_3x_3,\\
[\omega_{k_1,k_2,k_3}] &= (k_1 - ak_2 + ack_3)x_1 + (k_2 - ck_3)y_2 + k_3y_3.
\end{aligned}$$

The positivity of the coefficients in these equations is equivalent to $k_1 - |a|(k_2 - |c|k_3) > 0,\ k_2 - |c|k_3 > 0,\ k_3 > 0$. This completes the proof. □

Example 4.1. Let us consider a specific case. Here (a,b,c) are one-half the (a,b,c) in the itemized list above. We take $(k_1,k_2,k_3) = (5,2,1)$ and ask which Bott manifolds belong to the symplectic manifold $\big((S^2)^3, \omega_{5,2,1})\big)$. Applying Theorem 4.6 we see that if $c = 0$ we have one constraint, namely $5 - 2|a| - |b| > 0$. Then using Example 1.6 we can take both a and b to be non-negative. So we have the possibilities $a = 2, b = 0$, $a = 1,\ b = 2,1,0$, and $a = 0,\ b = 4,3,2,1,0$. These give $M_3(4,0,0)$, $M_3(2,4,0)$, $M_3(2,2,0)$, $M_3(2,0,0)$, $M_3(0,8,0)$, $M_3(0,6,0)$, $M_3(0,4,0)$, $M_3(0,2,0)$, $M_3(0,0,0)$.

Now if $c \neq 0$ the constraint $2 - |c| > 0$ implies $c = \pm 1$ and we up to equivalence we can take $c = 1$. Then Theorem 4.6 gives the constraint $0 < 5 - |a|(2 - |c|) = 5 - |a|$.

Again using the equivalences of Example 1.6 we have $a = 0, 1, 2, 3, 4$ which gives the Bott towers $M_3(0,0,2)$, $M_3(2,2,2)$, $M_3(4,4,2)$, $M_3(6,6,2)$, $M_3(8,8,2)$. So there are 14 distinct equivalence classes of Bott towers that are compatible with the symplectic manifold $\big((S^2)^3, \omega_{5,2,1})\big)$, namely

$$M_3(0,0,0), M_3(0,2,0), M_3(0,4,0), M_3(0,6,0), M_3(0,8,0), M_3(2,0,0), M_3(4,0,0),$$
$$M_3(2,4,0), M_3(2,2,0), M_3(0,0,2), M_3(2,2,2), M_3(4,4,2), M_3(6,6,2), M_3(8,8,2).$$

Notice that there are nine Bott manifolds with cotwist ≤ 1, six with twist ≤ 1, five of which also have cotwist ≤ 1. Finally, there are four with twist and cotwist 2. By Theorem 4.3 these correspond to distinct conjugacy classes of maximal tori in the symplectomorphism group.

Using Theorem 4.6 we obtain a formula for the number $N_B(k_1, k_2, k_3)$ of Bott manifolds $M_3(2a, 2b, 2c)$ compatible with a given symplectic structure $\big((S^2)^3, \omega_{k_1,k_2,k_3}\big)$ as well as a growth estimate. For the growth estimates we put $k_3 = 1$ since clearly the growth slows for larger k_3.

Proposition 4.7. *The number of Bott manifolds compatible with the symplectic manifold* $\big((S^2)^3, \omega_{k_1,k_2,k_3}\big)$ *is*

$$\begin{aligned} N_B(k_1,k_2,k_3) &= N_{B,0}(k_1,k_2,k_3) + N_{B,\neq 0}(k_1,k_2,k_3) \\ &= \sum_{j=0}^{b_{max}} \left\lceil \frac{k_1 - jk_3}{k_2} \right\rceil + \sum_{j=1}^{c_{max}} \left\lceil \frac{k_1}{k_2 - jk_3} \right\rceil . \end{aligned}$$

Furthermore, when k_1, k_2, k_3 *are positive integers we have the growth estimates*

$$\frac{k_1(k_1-1)}{2k_2} \leq N_{B,0}(k_1,k_2,1) \leq \frac{k_1(k_1-1)}{2k_2} + \frac{(k_1-1)(k_2+1)}{k_2},$$

and when $c_{max} > 0$

$$\begin{aligned} &k_1(\ln(k_2-1) + \gamma + \epsilon_{k_2-1}) \\ &\quad \leq N_{B,\neq 0}(k_1,k_2,1) \leq k_1(\ln(k_2-1) + \gamma + \epsilon_{k_2-1}) + k_2 - 1 + \ln(k_2-1) + \gamma + \epsilon_{k_2-1} \end{aligned}$$

where $\gamma \approx .577$ *is Euler's constant,* $\epsilon_1 = 1 - \gamma$ *and* ϵ_k *goes to* 0 *as* $1/2k$.

Proof. According to Theorem 4.6 there are two cases to consider, $c = 0$ and $c \neq 0$. If $c = 0$ we can count the number of nonnegative integers a in item (1) of Theorem 4.6 that contribute to each $b = 0, \ldots, b_{max}$ where b_{max} is the largest nonnegative integer b such that $k_1 - bk_3 > 0$. This gives $N_B(k_1, k_2, k_3)$ when $c = 0$,

$$N_{B,0}(k_1,k_2,k_3) = \sum_{j=0}^{b_{max}} \left\lceil \frac{k_1 - jk_3}{k_2} \right\rceil$$

where $\lceil \cdot \rceil$ is the ceiling function. Similarly, in the case $c > 0$ we count the number of nonnegative integers a in item (2) of Theorem 4.6 that contribute to each $c = 0, \ldots, c_{max}$ where c_{max} is the largest nonnegative integer c such that $k_2 - ck_3 > 0$. This gives the number of compatible Bott manifolds for each $c \neq 0$, namely

$$N_{B,c}(k_1,k_2,k_3) = \left\lceil \frac{k_1}{k_2 - ck_3} \right\rceil$$

which implies the formula.

We prove the second growth estimate and leave the first to the reader. First we have

$$N_{B,\neq 0}(k_1,k_2,1)=\sum_{j=1}^{k_2-1}\left\lceil\frac{k_1}{k_2-j}\right\rceil\geq\sum_{j=1}^{k_2-1}\frac{k_1}{k_2-j}=k_1\sum_{m=1}^{k_2-1}\frac{1}{m}=k_1(\ln(k_2-1)+\gamma+\epsilon_{k_2-1}).$$

On the other hand using the well known relation between floor and ceiling functions we find

$$\begin{aligned}N_{B,\neq 0}(k_1,k_2,1)&=\sum_{j=1}^{k_2-1}\left\lceil\frac{k_1}{j}\right\rceil=\sum_{j=1}^{k_2-1}\left(\left\lfloor\frac{k_1+1}{j}\right\rfloor+1\right)\leq(k_2-1)+(k_1+1)\sum_{j=1}^{k_2-1}\frac{1}{j}\\&=(k_2-1)+(k_1+1)(\ln(k_2-1)+\gamma+\epsilon_{k_2-1})\end{aligned}$$

which implies the estimate. □

An easy corollary of Theorem 4.3 and Proposition 4.7 is

Corollary 4.8. *The number of conjugacy classes of maximal tori of dimension n in the symplectomorphism group* $\mathrm{Symp}((S^2)^3,\omega_{k_1,k_2,k_3})$ *is at least*

$$\sum_{j=0}^{b_{max}}\left\lceil\frac{k_1-jk_3}{k_2}\right\rceil+\sum_{j=1}^{c_{max}}\left\lceil\frac{k_1}{k_2-jk_3}\right\rceil.$$

Note that the inequality $k_1+ack_3-ak_2>0$ implies that the number $N_{B,\neq 0}(k_1,k_2,k_3)$ grows as the product ac. In fact, when $c_{max}>0$ the largest values of a,b,c occur when $k_3=1$ in which case we find the Bott tower $M_3\big(2(k_1-1),2(k_1-1)(k_2-1),2(k_2-1)\big)$.

When $c_{max}=k_2-1>0$ the number $N_{B,\neq 0}(k_1,k_2,1)$ grows like $k_1\ln(k_2-1)$. Notice that the error term $k_2-1+\ln(k_2-1)+\gamma+\epsilon_{k_2-1}$ is independent of k_1 and grows linearly with k_2.

Example 4.2. Consider the symplectic manifold $\big((S^2)^3,\omega_{11,6,1}\big)$. Here $N_{B,0}(11,6,1)=16$ and $N_{B,\neq 0}(11,6,1)=27$, giving $N_B(11,6,1)=43$. It is also a straightforward process to delineate the Bott manifolds. We see in the case of $c=0$ we have the 17 Bott manifolds $M_3(0,2b,0):b\in\{0,\dots,10\}$ and $M_3(2,2b,0):b\in\{0,\dots,5\}$; whereas, for $c\neq 0$ we have the 27 Bott manifolds

$$M_3(2a,10a,10):a\in\{0,\dots,10\},M_3(2a,8a,8):a\in\{0,\dots,5\},$$
$$M_3(2a,6a,6):a\in\{0,1,2,3\},M_3(2a,4a,4):a\in\{0,1,2\},M_3(2a,2a,2):a\in\{0,1,2\}.$$

The largest values of a,b,c occur for the Bott manifold $M_3(20,100,10)$.

4.3. **Further symplectic equivalences.**

Example 4.3. Let us consider the Bott towers $M_{N+1}(\mathbf{k})$ of twist ≤ 1. To begin with we fix a symplectic (actually Kähler) structure by fixing $k_i<0$. In this case we know from Example 3.4 that the Kähler cone is the entire first orthant with respect to the basis $x_1,\dots,x_{N+1}$. Thus, there is a symplectic (Kähler) form $\omega_{\mathbf{r}}$ whose class is $[\omega_{\mathbf{r}}]=r_1x_1+\cdots+r_{N+1}x_{N+1}$. We denote this symplectic manifold by $(M_{N+1}(\mathbf{k}),\omega_{\mathbf{r}})$, and for simplicity we fix an order of the k_i and consider the case that they all are distinct. Then for $N>2$ by Theorem 2.7 there are at most 2^{N-1} distinct Bott manifolds, including $(M_{N+1}(\mathbf{k}),\omega_{\mathbf{r}})$ itself that are compatible with $(M_{N+1}(\mathbf{k}),\omega_{\mathbf{r}})$ as a symplectic

manifold which are obtained by reversing the sign of k_i for each $i = 1, \dots, N$ taking into account that $M_{N+1}(-\mathbf{k})$ is equivalent to $M_{N+1}(\mathbf{k})$. We now choose a diffeomorphic Bott manifold $M_{N+1}(\mathbf{k}')$ by reversing the signs of some of the k_i. As in Example 3.4 by a choice of order we can assume that $k'_i = k_i < 0$ for $i = 1, \dots, m$ and $k'_{i+1} = -k_{i+1} > 0$ for $i = m+1, \dots, N$. Then choosing $D_{N+1} = D_{v_{N+1}}$ the Kähler cone is given by $r_i > -k_i r_{N+1}$ for $i = 1, \dots, m$, and $r_i > 0$ for $j = m+1, \dots, N$. We have arrived at

Proposition 4.9. *The Bott manifold $M_{N+1}(\mathbf{k}')$ with $k'_i = k_i < 0$ for $i = 1, \dots, m$ and $k'_{i+1} = -k_{i+1} > 0$ for $i = m+1, \dots, N$ is compatible with the symplectic manifold $(M_{N+1}(\mathbf{k}), \omega_{\mathbf{r}})$ if and only if $r_i > -k_i r_{N+1}$ for $i = 1, \dots, m$, and $r_i > 0$ for $j = m+1, \dots, N$. Thus, if the set $\{r_i\}$ satisfies these inequalities and if $N > 2$ generically we have $|\mathcal{B}_n(M_{N+1}(\mathbf{k}), \omega_{\mathbf{r}})| = 2^{N-1}$.*

One can check that the equivalent Bott tower $M_{N+1}(-\mathbf{k}')$ which swaps $D_{v_{N+1}}$ and $D_{u_{N+1}}$ gives the same data. Proposition 4.9 shows that when the components of $\mathbf{k}$ have different signs generally there are constraints for a Bott manifold to be compatible with a given symplectic structure. For example, a symplectic structure ω whose class is $x_1 + \cdots + x_{N+1}$ is only compatible with the Bott towers $M_{N+1}(\mathbf{k})$ with either $k_i < 0$ for all i, or $k_i > 0$ for all i.

5. The generalized Calabi and admissible constructions

The purpose of this section is to obtain existence results for extremal Kähler metrics on Bott manifolds. We use the generalized Calabi construction to prove the general result of Theorem 1 (2), and the admissible construction to prove existence of extremal Kähler metrics for some of our running examples. We begin with a brief outline of these two constructions.

5.1. The generalized Calabi construction.

The *generalized Calabi construction* was introduced in [ACGT04, Section 2.5] and further discussed in [ACGT11, Section 2.3]. This construction can be used to obtain existence results for extremal Kähler metrics and in particular it will be used to prove Theorem 1 (2). In order to make this paper self-contained we review the generalized Calabi construction for the case of "no blowdowns" and where the resulting manifold is a bundle over a compact Riemann surface Σ with fiber equal to a toric Kähler manifold.

Definition 5.1. *Generalized Calabi data* of dimension $\ell + 1$ and rank ℓ consist of the following.

(1) A compact Riemann surface Σ with Kähler structure $(\omega_\Sigma, g_\Sigma)$. For simplicity we assume that g_Σ has constant scalar curvature $Scal_\Sigma$ and $\omega_\Sigma/2\pi$ is primitive. Hence $Scal_\Sigma = 4(1-\mathfrak{g})$, where $\mathfrak{g}$ is the genus of Σ. If ρ_Σ denotes the Ricci form then we have $\rho_\Sigma = 2(1-\mathfrak{g})\omega_\Sigma$.
(2) A compact toric ℓ-dim Kähler manifold (V, g_V, ω_V) with Delzant polytope $\Delta \subseteq \mathfrak{t}^*$ and momentum map $z\colon V \to \Delta$.
(3) A principal T^ℓ bundle, $P \to \Sigma$, with a principal connection of curvature $\omega_\Sigma \otimes p \in \Omega^{1,1}(\Sigma, \mathfrak{t})$, where T^ℓ is the ℓ-torus acting on V and $p \in \mathfrak{t}$.
(4) A constant $p_0 \in \mathbb{R}$ such that the $(1,1)$-form $p_0\,\omega_\Sigma + \langle v, \omega_\Sigma \otimes p\rangle$ is positive for $v \in \Delta$.

From this data we may define the manifold

$$M = P \times_{T^\ell} V = \mathring{M} \times_{(T^\ell)^{\mathbb{C}}} V \to \Sigma,$$

where $\mathring{M} = P \times_{T^\ell} z^{-1}(\mathring{\Delta})$ and $\mathring{\Delta}$ is the interior of Δ. Since the curvature 2-form of P has type $(1,1)$, $\mathring{M}$ is a holomorphic principal $(T^\ell)^{\mathbb{C}}$ bundle with connection $\theta \in \Omega^1(\mathring{M}, \mathfrak{t})$ and M is a complex manifold.

On $\mathring{M}$ we define Kähler structures of the form

$$\begin{aligned} g &= (p_0 + \langle p, z\rangle)\, g_{\mathbb{CP}^1} + \langle dz, \mathbf{G}, dz\rangle + \langle \theta, \mathbf{H}, \theta\rangle \\ \omega &= (p_0 + \langle p, z\rangle)\, \omega_{\mathbb{CP}^1} + \langle dz \wedge \theta\rangle \\ d\theta &= \omega_{\mathbb{CP}^1} \otimes p, \end{aligned} \tag{55}$$

where $\mathbf{G} = \mathrm{Hess}(U) = \mathbf{H}^{-1}$, U is the symplectic potential [Gui94] of the chosen toric Kähler structure g_V on V, and $\langle \cdot, \cdot, \cdot\rangle$ denotes the pointwise contraction $\mathfrak{t}^* \times S^2\mathfrak{t} \times \mathfrak{t}^* \to \mathbb{R}$ or the dual contraction.

The *generalized Calabi construction* arises from seeing (55) as a blueprint for the construction of various Kähler metrics on M by choosing various $S^2\mathfrak{t}^*$-valued functions $\mathbf{H}$ on $\mathring{\Delta}$ satisfying that

- [integrability] $\mathbf{H} = \mathrm{Hess}(U)^{-1}$ for some smooth function U on Δ^0.
- [smoothness] $\mathbf{H}$ is the restriction to $\mathring{\Delta}$ of a smooth $S^2\mathfrak{t}^*$-valued function on Δ;
- [boundary values] for any point z on the codimension one face $F_i \subseteq \Delta$ with inward normal u_i, we have

$$\mathbf{H}_z(u_i, \cdot) = 0 \qquad \text{and} \qquad (d\mathbf{H})_z(u_i, u_i) = 2u_i, \tag{56}$$

  where the differential $d\mathbf{H}$ is viewed as a smooth $S^2\mathfrak{t}^* \otimes \mathfrak{t}$-valued function on Δ;
- [positivity] for any point z in the interior of a face $F \subseteq \Delta$, $\mathbf{H}_z(\cdot, \cdot)$ is positive definite when viewed as a smooth function with values in $S^2(\mathfrak{t}/\mathfrak{t}_F)^*$.

Then (55) extends to a Kähler metric on M. Metrics constructed this way are called *compatible Kähler metrics* and their Kähler classes are called *compatible Kähler classes*.

Remark 5.2. The toric bundle M equipped with a compatible Kähler metric as above is a so-called *rigid, semisimple* toric bundle over $\mathbb{CP}^1$ [ACGT11]. We also use this terminology for any toric bundle $M = P \times_{T^\ell} V$ where P and V are as defined above.

Remark 5.3. From now on we assume that $\Sigma = \mathbb{CP}^1$, so $Scal_\Sigma = Scal_{\mathbb{CP}^1} = 4$. Note that, since $(g_{\mathbb{CP}^1}, \omega_{\mathbb{CP}^1})$ is the toric Fubini–Study Kähler structure, the Kähler metrics arising this way are themselves toric and generalize the notion of Calabi toric metrics from [Leg11].

It follows from [ACGT11, Theorem 3 & Remark 6] that if V admits a toric extremal Kähler metric g_V, then M admits compatible extremal Kähler metrics (at least in some Kähler classes). We now use this result inductively to prove Theorem 1 (2).

Proof of Theorem 1 (2). First we note that for stage 2 Bott manifolds, the result holds by Calabi's original construction of extremal metrics on Hirzebruch surfaces [Cal82]. Now consider the Bott tower

$$M_{k+1} \xrightarrow{\pi_{k+1}} M_k \xrightarrow{\pi_k} \cdots M_2 \xrightarrow{\pi_2} M_1 = \mathbb{CP}^1$$

described by the matrix A_{k+1}. By construction M_{k+1} is a rigid semisimple toric bundle over $\mathbb{CP}^1$. Now assume by induction that for a fixed k and for each $k \times k$ matrix A of the form (3) (with n replaced by k) the corresponding Bott manifold M_k has a compatible extremal Kähler metric. We have seen that the fiber of the map $M_{k+1} \xrightarrow{\pi_{k+1} \circ \cdots \circ \pi_2} \mathbb{CP}^1$ is the Bott manifold $\tilde{M}_k$ obtained by removing the first row and column of the matrix defining M_{k+1}. We can now apply [ACGT11, Theorem 3 & Remark 6] to conclude that M_{k+1} admits a compatible toric extremal Kähler metric in some Kähler class. □

5.2. **The admissible construction.** When V is $\mathbb{CP}^1$ in Definition 5.1, we have Calabi's construction of extremal Kähler metrics on Hirzebruch surfaces [Cal82]. This construction is a special case of the so-called *admissible construction* [ACGT08a]. Even though the admissible construction as summarized below (again for "no blowdowns") is not universally useful for Bott manifolds, there are still some interesting explicit examples occurring in the literature [KS86, Hwa94, Gua95] which we exhibit below in the context of Bott manifolds. These provide some further existence results for extremal Kähler metrics on Bott manifolds with twist ≤ 2.

Let S be a compact complex manifold admitting a local Kähler product metric, whose components are Kähler metrics denoted $(\pm g_a, \pm\omega_a)$, and indexed by $a \in \mathcal{A} \subseteq \mathbb{Z}^+$. Here $(\pm g_a, \pm\omega_a)$ is the Kähler structure. In this notation we allow for the tensors g_a to possibly be negatively definite—a parametrization given later justifies this convention. Note that in all our applications, each $\pm g_a$ will be CSC. The real dimension of each component is denoted $2d_a$, while the scalar curvature of $\pm g_a$ is given as $\pm 2d_a s_a$. Next, let $\mathcal{L}$ be a hermitian holomorphic line bundle over S, such that $c_1(\mathcal{L}) = \sum_{a\in\mathcal{A}} [\omega_a/2\pi]$, and let $\mathbb{1}$ denote the trivial line bundle over S. Then, following [ACGT08a], the total space of the projectivization $M = P(\mathbb{1} \oplus \mathcal{L}) \to S$ is called an *admissible manifold*.

Let M be an admissible manifold of complex dimension d. An *admissible Kähler metric* on M is a Kähler metric constructed as follows. Consider the circle action on M induced by the standard circle action on $\mathcal{L}$. It extends to a holomorphic $\mathbb{C}^*$ action. The open and dense set M_0 of stable points with respect to the latter action has the structure of a principal circle bundle over the stable quotient. The hermitian norm on the fibers induces via a Legendre transform a function $z\colon M_0 \to (-1, 1)$ whose extension to M has critical submanifolds $z^{-1}(1) = P(\mathbb{1} \oplus 0)$ and $z^{-1}(-1) = P(0 \oplus \mathcal{L})$. Letting θ be a connection one form for the Hermitian metric on M_0, with curvature $d\theta = \sum_{a\in\mathcal{A}} \omega_a$, an admissible Kähler metric and form are given up to scale by the respective formulas

$$g = \sum_{a\in\mathcal{A}} \frac{1 + r_a z}{r_a} g_a + \frac{dz^2}{\Theta(z)} + \Theta(z)\theta^2, \quad \omega = \sum_{a\in\mathcal{A}} \frac{1 + r_a z}{r_a} \omega_a + dz \wedge \theta, \tag{57}$$

valid on M_0. Here Θ is a smooth function with domain containing $(-1, 1)$ and r_a, $a \in \mathcal{A}$ are real numbers of the same sign as g_a and satisfying $0 < |r_a| < 1$. The complex structure yielding this Kähler structure is given by the pullback of the base complex structure along with the requirement $J dz = \Theta\theta$. The function z is a hamiltonian for the Killing vector field $K = J \operatorname{grad} z$, hence a momentum map for the circle action, so M decomposes into the free orbits $M_0 = z^{-1}((-1, 1))$ and the special orbits $z^{-1}(\pm 1)$. Finally, θ satisfies $\theta(K) = 1$.

For an admissible metric (57), the 2-form

$$\phi = \sum_{a\in\mathcal{A}} -\frac{1+r_a z}{r_a^2}\omega_a + z dz\wedge\theta$$

is a *hamiltonian* 2*-form* in the sense that

$$\nabla_X \phi = \frac{1}{2}(d\,\mathrm{tr}_\omega\,\phi\wedge JX - d^c\,\mathrm{tr}_\omega\,\phi\wedge X)$$

for any vector field X, where $\mathrm{tr}_\omega\,\phi = \langle\phi,\omega\rangle$ is the trace with respect to ω. The theory of [ACG06] implies that the metrics (57) are (up to scale) the general form of Kähler metrics admitting a hamiltonian 2-form of order 1.

For such g to define a metric on M, Θ must satisfy

$$\text{(i) } \Theta(z) > 0, \quad -1 < z < 1, \quad \text{(ii) } \Theta(\pm 1) = 0, \quad \text{(iii) } \Theta'(\pm 1) = \mp 2. \tag{58}$$

where (ii) and (iii) are necessary and sufficient for g to extend to M, while (i) ensures positive definiteness.

The Kähler class $\Omega_r = [\omega]$ of an admissible metric is also called *admissible* and is uniquely determined by the parameters $r_a : a \in \mathcal{A}$, once the data associated with M (i.e., d_a, s_a, g_a etc.) is fixed. The $r_a : a \in \mathcal{A}$, together with the data associated with M is called *admissible data*—see [ACGT08a, Section 1] for further background on this set-up. Note that different choices of Θ (with all else being fixed) define different compatible complex structures with respect to the same symplectic form ω. However, as is discussed in [ACGT08a], up to a fiber preserving S^1-equivariant diffeomorphism, we may consider the complex structure fixed; then functions Θ satisfying (58) determine Kähler metrics within the same Kähler class.

It is useful to define a function $F(z)$ by the formula $\Theta(z) = F(z)/p_c(z)$, where $p_c(z) = \prod_{a\in\mathcal{A}}(1+r_a z)^{d_a}$. With this notation g has scalar curvature

$$Scal_g = \sum_{a\in\mathcal{A}} \frac{2d_a s_a r_a}{1+r_a z} - \frac{F''(z)}{p_c(z)} \tag{59}$$

One may now check [ACG06, ACGT08a] that with fixed admissible data, there is precisely one function, the *extremal polynomial* $F(z)$, so that the right hand side of (59) is an affine function of z and the corresponding $\Theta(z)$ satisfies (ii) and (iii) of (58). This means that as long as also (i) of (58) is satisfied, i.e., the extremal polynomial is positive for $z \in (-1,1)$, we would have an admissible extremal metric (and CSC if the affine function is constant). However, there is in general no guarantee that the extremal polynomial is positive for $z \in (-1,1)$. In the special case where $Scal_{\pm g_a}$ is non-negative, results of Hwang [Hwa94] and Guan [Gua95] show that positivity is satisfied and thus every admissible Kähler class has an admissible extremal metric determined by its *extremal polynomial* $F(z)$.

5.3. **Applications of the admissible construction to Bott manifolds.** We begin by considering the Bott towers $M_{N+1}(\mathbf{k})$ of twist ≤ 1 first described in Example 1.1, which are $\mathbb{CP}^1$ bundles over the product complex manifold $(\mathbb{CP}^1)^N$, realized as the projectavization $\mathbb{P}(\mathbb{1}\oplus\mathcal{O}(k_1,k_2,\ldots,k_N)) \to (\mathbb{CP}^1)^N$. Making contact with the admissible construction we note that $\mathcal{A} = \{1,2,\ldots,N\}$, i.e., for $a \in \{1,2,\ldots,N\}$, $\pm g_a$ is the Fubini–Study metric on $\mathbb{CP}^1$ with positive constant scalar curvature $\pm 2s_a = \pm 4/k_a$ for

some non-zero integer k_a. By the discussion above, every admissible Kähler class has an admissible extremal metric and if there is at least one pair $i, j \in \{1, 2, \ldots, N\}$ such that $k_i\, k_j < 0$ then, by [Hwa94, Corollary 1.2 & Theorem 2], some of these classes admit admissible CSC metrics. Moreover, since in this case every Kähler class is admissible (see [ACGT08a, Remark 2]), we may conclude as follows.

Proposition 5.4. *Any stage $N+1$ Bott manifold $M_{N+1}(\mathbf{k})$, with matrix of the form* (5) *such that $\prod_{a=1}^{N} k_a \neq 0$, admits an admissible extremal Kähler metric in every Kähler class. Moreover, when the $k_1, \ldots, k_N$ do not all have the same sign, some of these metrics are CSC.*

Now, if we let x_a denote the pullback from the ath factor of $\mathbb{CP}^1 \times \cdots \times \mathbb{CP}^1$ of the generator of $H^2(\mathbb{CP}^1, \mathbb{Z})$, i.e., $x_a = \frac{[\omega_a]}{2\pi k_a}$, then the discussion in [ACGT08a, Section 1.4] allows us to write the (admissible) Kähler classes as

$$\Omega_r = 2\pi \left[2x_{N+1} + \sum_{a=1}^{N} k_a(1 + 1/r_a)x_a \right],$$

where x_{N+1} is defined to be $PD(D^{\infty}_{N+1})$ as usual.

By the famous result of Koiso and Sakane [KS86], when $N = 2m$,

$$k_a = \begin{cases} 1 & \text{if} \quad 1 \leq a \leq m \\ -1 & \text{if} \quad m+1 \leq a \leq 2m \end{cases}$$

and $r_a = k_a/2$, the admissible extremal metric in Ω_r is Kähler–Einstein. More generally, when $N = 2m$,

$$k_a = \begin{cases} q & \text{if} \quad 1 \leq a \leq m \\ -q & \text{if} \quad m+1 \leq a \leq 2m \end{cases}$$

for some $q \in \mathbb{Z}^+$, and

$$r_a = \begin{cases} r & \text{if} \quad 1 \leq a \leq m \\ -r & \text{if} \quad m+1 \leq a \leq 2m \end{cases}$$

for some $r \in (0, 1)$, the admissible extremal metric in Ω_r is CSC with positive scalar curvature.

Suppose for example that $r = 1/k$ for $k \in \mathbb{Z}^+ \setminus \{1\}$ in the above. Then

$$\Omega_r/(2\pi) = \left[2x_{2m+1} + q(k+1) \sum_{a=1}^{m} x_a + q(k-1) \sum_{a=m+1}^{2m} x_a \right]$$

is an integer Kähler class and hence defines a line bundle. If we take the admissible CSC representative of Ω_r on the admissible stage $2m+1$ Bott manifold

$$M_{2m+1} = \mathbb{P}(\mathbb{1} \oplus \mathcal{O}(\overbrace{q, \ldots, q}^{m}, \overbrace{-q, \ldots, -q}^{m}) \to \overbrace{\mathbb{CP}^1 \times \cdots \times \mathbb{CP}^1}^{2m}$$

and feed it into another admissible construction, with M_{2m+1} now playing the role of the base S, we get explicit extremal admissible metrics on the resulting stage $2m+2$

Bott manifold $M_{2m+2} = \mathbb{P}(\mathbb{1} \oplus \mathcal{L}_{2m+2}) \to M_{2m+1}$ in all Kähler classes spanned by

$$\left(1 + \frac{1}{r_{2m+1}}\right)\left[2x_{2m+1} + q(k+1)\sum_{a=1}^{m} x_a + q(k-1)\sum_{a=m+1}^{2m} x_a\right] + 2x_{2m+2},$$

where $r_{2m+1} \in (0,1)$ and $x_{2m+2} = PD(D^{\infty}_{2m+2})$. Thus we may conclude as follows.

Proposition 5.5. *Assume that $q \in \mathbb{Z}^+$ and $k \in \mathbb{Z}^+ \setminus \{1\}$ and consider the height $2m+2$ Bott tower $M_n(A)$ with*

$$A = \begin{pmatrix} \mathbb{1}_m & 0_m & \cdots & 0 \\ 0_m & \mathbb{1}_m & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ q\mathbf{1}_m & -q\mathbf{1}_m & 1 & 0 \\ q(k+1)\mathbf{1}_m & q(k-1)\mathbf{1}_m & 2 & 1 \end{pmatrix}$$

Then there is a 2-dimensional subcone of the Kähler cone of $M_n(A)$ whose elements admit admissible extremal metrics. Furthermore, the stage $2m+1$ Bott manifold obtained by removing the last column and the last row of A admits admissible extremal Kähler metrics in every Kähler class of the entire Kähler cone and admissible CSC Kähler metrics in some Kähler classes. If $q = 1$ this stage $2m+1$ Bott manifold also admits a Kähler–Einstein metric.

(*Here admissibility is with respect the $\mathbb{CP}^1$-fibration of $M_n(A)$ over the previous stage $M_{n-1}(A)$ in the Bott tower.*)

Remark 5.6. A similar result to Proposition 5.4 holds in the degenerate case when some of the components of $\mathbf{k}$ vanish. These Bott towers as in (17) have the form

$$M_{N+1}(\mathbf{k}) \cong M_m(\tilde{\mathbf{k}}) \times \overbrace{\mathbb{CP}^1 \times \cdots \times \mathbb{CP}^1}^{N+1-m} \tag{60}$$

where none of the components $k_1, \ldots, k_{m-1}$ of $\tilde{\mathbf{k}}$ vanish.

5.4. **Relation with c-projective geometry.** We now present a curious observation in the case of the stage 3 Bott manifold of the Bott tower with matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}.$$

That is, we take a closer look at admissible CSC Kähler metrics on $M_3(0,1,-1) = \mathbb{P}(\mathbb{1} \oplus \mathcal{O}(-1,1)) \to \mathbb{CP}^1 \times \mathbb{CP}^1$.

It turns out that on this special case of a stage 3 Bott manifold, there exist infinitely many pairs of c-projectively equivalent CSC Kähler metrics which are not affinely equivalent. For background on c-projective geometry we refer the reader to [CEMN16]. Here we focus on the c-projective equivalent admissible metrics.

Let (g, ω, J) be a Kähler metric as in (57) and for $\alpha, \beta \in \mathbb{R}$ consider the hamiltonian 2-form

$$\phi_{\alpha,\beta} := (\alpha\phi + \beta\omega) = \sum_{a \in \mathcal{A}} \frac{(\beta r_a - \alpha)(1 + r_a z)}{r_a^2}\omega_a + (\alpha z + \beta)dz \wedge \theta.$$

According to the general theory [CEMN16], we can form a symmetric and J-invariant tensor $g_{\alpha,\beta}^{-1}\colon T^*M \to TM$ by

$$g_{\alpha,\beta}^{-1} = (\det A)^{1/2} A \circ g^{-1},$$

where $A\colon TM \to TM$ is the g-selfadjoint tensor defined by

$$\phi_{\alpha,\beta}(\cdot, J\cdot) = g(A\cdot, \cdot).$$

If $g_{\alpha,\beta}^{-1}$ is invertible, then $g_{\alpha,\beta}$ defines a Kähler metric $(g_{\alpha,\beta}, J, \hat{\omega})$ such that g and $\hat{g}$ are c-projectively equivalent. In that case, $(g_{\alpha,\beta}, J, \hat{\omega})$ also admits a hamiltonian 2-form (of same order) and so should be admissible up to scale and in appropriate coordinates.

Now we see that

$$(\alpha\phi + \beta\omega)(\cdot, J\cdot) = \sum_{a\in\mathcal{A}} \frac{(\beta r_a - \alpha)(1 + r_a z)}{r_a^2} g_a + (\alpha z + \beta)\Big(\frac{dz^2}{\Theta(z)} + \Theta(z)\theta^2\Big).$$

If we (locally) extend dz, θ to a basis of 1-forms by pullbacks from the base, then we have corresponding dual vector fields Z, K and may write

$$g^{-1} = \sum_{a\in\mathcal{A}} \frac{r_a}{1 + r_a z} g_a^{-1} + \Theta(z) Z^2 + \frac{1}{\Theta(z)} K^2.$$

Now $A\, g^{-1}$ must be equal to

$$(\alpha\phi + \beta\omega)(\cdot, J\cdot)^\sharp = \sum_{a\in\mathcal{A}} \frac{(\beta r_a - \alpha)}{(1 + r_a z)} g_a^{-1} + (\alpha z + \beta)(\Theta(z) Z^2 + \frac{1}{\Theta(z)} K^2).$$

Therefore

$$\det(A)^{\frac{1}{2}} = \frac{\det(\alpha\phi + \beta\omega)(\cdot, J\cdot)^\sharp)^{\frac{1}{2}}}{\det(g^{-1})^{\frac{1}{2}}} = \frac{(\alpha z + \beta) \prod_{a\in\mathcal{A}} (\beta r_a - \alpha)^{d_a}}{\prod_{a\in\mathcal{A}} r_a^{d_a}}$$

and so (since $g_{\alpha,\beta}^{-1} = \det(A)^{\frac{1}{2}} A\, g^{-1}$) we have (for appropriate choices of α and β) that the metric $g_{\alpha,\beta}$ given up to scale by

$$\frac{1}{(\alpha z + \beta)} \left(\sum_{a\in\mathcal{A}} \frac{(1 + r_a z)}{(\beta r_a - \alpha)} g_a + (\alpha z + \beta)^{-1} \left(\frac{dz^2}{\Theta(z)} + \Theta(z)\theta^2 \right) \right) \tag{61}$$

Now consider the change of coordinates

$$\tilde{z} = \frac{\alpha + \beta z}{\alpha z + \beta}, \quad z = \frac{\alpha - \beta\tilde{z}}{\alpha\tilde{z} - \beta}, \quad dz = \frac{\beta^2 - \alpha^2}{(\alpha\tilde{z} - \beta)^2}\, d\tilde{z}.$$

Then $\tilde{z} \in [-1, 1]$ if and only if $z \in [-1, 1]$ and moreover

$$1 + r_a z = \frac{(\alpha - \beta r_a)\tilde{z} + \alpha r_a - \beta}{\alpha\tilde{z} - \beta}.$$

We now rewrite (61) according to the change of coordinates and get

$$\frac{1}{(\beta^2 - \alpha^2)} \left(\sum_{a\in\mathcal{A}} \frac{((\beta r_a - \alpha)\tilde{z} + \beta - \alpha r_a)}{(\beta r_a - \alpha)} g_a + \frac{(\beta^2 - \alpha^2) d\tilde{z}^2}{(\alpha\tilde{z} - \beta)^2 \Theta(z)} + \frac{(\alpha\tilde{z} - \beta)^2 \Theta(z)}{(\beta^2 - \alpha^2)} \theta^2 \right).$$

Now, with

$$\Theta_{\alpha,\beta}(\tilde{z}) = \frac{(\alpha\tilde{z}-\beta)^2\Theta(z)}{(\beta^2-\alpha^2)} = \frac{d\tilde{z}}{dz}\Theta(z) \quad \text{and} \quad (r_a)_{\alpha,\beta} = \frac{\beta r_a - \alpha}{\beta - \alpha r_a},$$

we recognize this as an admissible metric.

It is easy to check that the end point conditions (58) are satisfied for $\Theta_{\alpha,\beta}(\tilde{z})$ at $\tilde{z} = \pm 1$ and the complex structure remains compatible with the admissible set-up:

$$Jd\tilde{z} = \frac{d\tilde{z}}{dz}Jdz = \frac{d\tilde{z}}{dz}\Theta(z)\theta = \Theta_{\alpha,\beta}(\tilde{z})\theta.$$

Instead of looking at

$$\Theta_{\alpha,\beta}(\tilde{z}) = \frac{(\alpha\tilde{z}-\beta)^2\Theta(z)}{(\beta^2-\alpha^2)}$$

we consider the corresponding change in $F(z)$ as defined in Section 5.3. From

$$F_{\alpha,\beta}(\tilde{z}) = \Theta_{\alpha,\beta}(\tilde{z})\prod_{a\in\mathcal{A}}(1+(r_a)_{\alpha,\beta}\tilde{z})^{d_a} \quad \text{and} \quad \Theta(z) = \frac{F(z)}{\prod_{a\in\mathcal{A}}(1+r_a z)^{d_a}}$$

it easily follows that

$$F_{\alpha,\beta}(\tilde{z}) = \frac{(\beta-\alpha\tilde{z})^{d+1}F(\frac{\alpha-\beta\tilde{z}}{\alpha\tilde{z}-\beta})}{(\beta^2-\alpha^2)\left(\prod_{a\in\mathcal{A}}(\beta-\alpha r_a)^{d_a}\right)}. \tag{62}$$

Remark 5.7. Note that $\langle\phi,\omega\rangle$ is not a constant, so as long as $\alpha \neq 0$; then g and $g_{\alpha,\beta}$ are not affinely equivalent.

We now consider admissible metrics on $P(\mathcal{O}\oplus\mathcal{O}(1,-1)) \to \mathbb{CP}^1\times\mathbb{CP}^1$ by choosing $\mathcal{A} = \{1,2\}$ with g_1 and $-g_2$ being Kähler–Einstein metrics, $s_1 = 2$, $s_2 = -2$, and $d_i = 1$. As discussed in Section 5.3, each (admissible) Kähler class, determined by $0 < r_1 < 1$ and $-1 < r_2 < 0$, admits an admissible extremal Kähler metric with extremal polynomial $F(z)$. Setting $q = 1$ in [ACGT08a, Theorem 9], $F(z)$ satisfies the CSC conditions if and only if $r_2 = -r_1$ or $r_2 = -1 + r_1$. In each of these families the Kähler–Einstein condition on $F(z)$ is satisfied precisely when $r_2 = -r_1 = -1/2$ which corresponds to the Koiso–Sakane Kähler–Einstein metric [KS86].

Remark 5.8. Note that we already discussed the $r_2 = -r_1$ solutions in Section 5.3 and saw that this family of solutions have higher dimensional analogues on

$$M_{2m+1} = \mathbb{P}(\mathbb{1}\oplus\mathcal{O}(\overbrace{1,\dots,1}^{m},\overbrace{-1,\dots,-1}^{m}) \to \overbrace{\mathbb{CP}^1\times\cdots\times\mathbb{CP}^1}^{2m}.$$

The second family of solutions, namely $r_2 = -1 + r_1$, also seems to have an analogue in higher dimensions. In the notation of Section 5.3 assume that Ω_r is the (admissible) Kähler class on M_{2m+1} given by

$$r_a = \begin{cases} r_+ & \text{if} \quad 1 \le a \le m \\ r_- & \text{if} \quad m+1 \le a \le 2m, \end{cases}$$

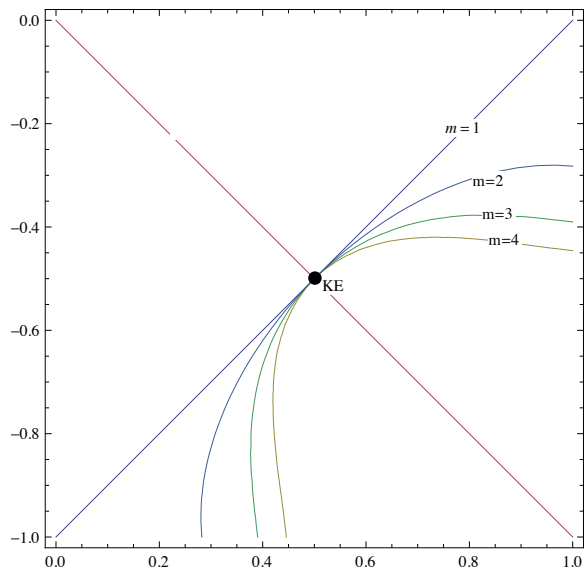

FIGURE 1. Behaviour of the second family for $1 \leq m \leq 4$.

where $0 < r_+ < 1$ and $-1 < r_- < 0$. Then, using [ACGT08a], we observe that Ω_r has an admissible CSC representative if and only if $\alpha_0\beta_1 - \alpha_1\beta_0 = 0$, where

$$\begin{aligned}
\alpha_0 &= \int_{-1}^{1} (1+r_+z)^m(1+r_-z)^m\,dz \\
\alpha_1 &= \int_{-1}^{1} z(1+r_+z)^m(1+r_-z)^m\,dz \\
\beta_0 &= (1+r_+)^m(1+r_-)^m + (1-r_+)^m(1-r_-)^m \\
&\qquad + 2m(r_+-r_-)\int_{-1}^{1}(1+r_+z)^{m-1}(1+r_-z)^{m-1}\,dz \\
\beta_1 &= (1+r_+)^m(1+r_-)^m - (1-r_+)^m(1-r_-)^m \\
&\qquad + 2m(r_+-r_-)\int_{-1}^{1} z(1+r_+z)^{m-1}(1+r_-z)^{m-1}\,dz.
\end{aligned}$$

For $m = 1$ this recovers the two families of solutions above. In general, for any positive integer m, it is easy to confirm the first family of solutions $r_- = -r_+$. Computer aided calculations for $m = 2, m = 3$, and $m = 4$ show a second family of solutions for these cases as well and we conjecture that this happens for any positive integer m. We have not been able to confirm this directly. Figure 1 shows the behavior of the second family for $m = 1, 2, 3$, and $m = 4$.

Returning to the $m = 1$ case, let us look at the second family of CSC Kähler metrics described above. We parametrize it by $0 < r < 1$ by setting $r_1 = r$ and $r_2 = -1 + r$. Let us denote this family by $\mathcal{K}S$. For a given choice of $0 < r < 1$, the CSC Kähler

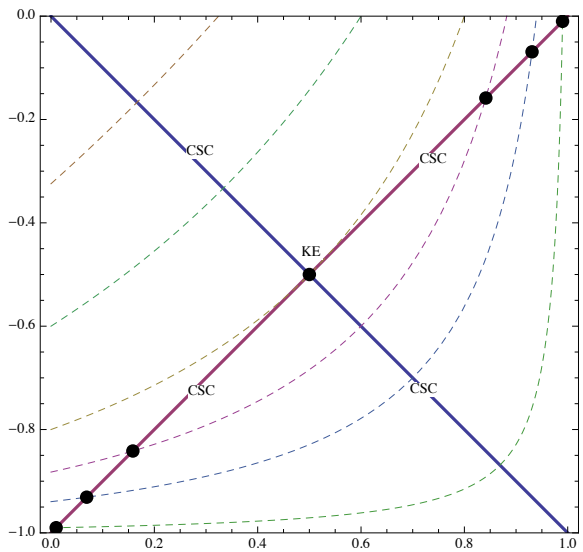

FIGURE 2. CSC Kähler classes.

metric corresponds to the extremal polynomial

$$\begin{aligned} F_r(z) &= (1-z^2)((1+r_1z)(1+r_2z)+\tfrac{1}{2}r_1r_2(1-z^2)) \\ &= \tfrac{1}{2}(1-z^2)(2-r+r^2+(4r-2)z+r(r-1)z^2). \end{aligned} \tag{63}$$

Now, we start with a CSC admissible metric $g \in \mathcal{K}S$ as above, determined by a parameter $0 < x < 1$. If we choose

$$\alpha = \frac{2r-1}{1-r+r^2} \quad \text{and} \quad \beta = 1$$

then is is not hard to check that the c-projectively equivalent metric $g_{\alpha,\beta}$ is defined. Further, since $r_1 = r$ and $r_2 = -1+r$, we calculate that, with this choice of α and β,

$$(r_1)_{\alpha,\beta} = \frac{\beta r - \alpha}{\beta - \alpha r} = 1 - r \quad \text{and} \quad (r_2)_{\alpha,\beta} = \frac{\beta(-1+r)-\alpha}{\beta-\alpha(-1+r)} = -r.$$

Thus the new metric $g_{\alpha,\beta}$ corresponds (up to scale) to a Kähler class of one of the metrics belonging to $\mathcal{K}S$ determined by the parameter $r_{\alpha,\beta} = 1-r$.

Figure 2 illustrates the Kähler cone (up to scaling) as parametrized by $0 < r_1 < 1$ and $-1 < r_2 < 0$. The two intersecting line segments correspond to the classes with CSC Kähler metrics and their intersection point is the Kähler–Einstein class. The line segment going from $(-1,0)$ to $(1,0)$ (not including the endpoints of course) correspond to the Kähler classes of the metrics in $\mathcal{K}S$. The trajectories represent how the Kähler classes change as we move in a c-projective class. The bold dots illustrate the above observation about c-projectively equivalent CSC metrics.

Now adapting (62) to our case we have

$$\begin{aligned} F_{\alpha,\beta}(\tilde{z}) &= \frac{(\beta-\alpha\tilde{z})^4 F_r(\frac{\alpha-\beta\tilde{z}}{\alpha\tilde{z}-\beta})}{(\beta^2-\alpha^2)\left((\beta-\alpha r_1)(\beta-\alpha r_2)\right)} = \frac{(\beta-\alpha\tilde{z})^4 F_r(\frac{\alpha-\beta\tilde{z}}{\alpha\tilde{z}-\beta})}{(\beta^2-\alpha^2)\left((\beta-\alpha r)(\beta-\alpha(-1+r))\right)} \\ &= \tfrac{1}{2}(1-\tilde{z}^2)(2-r+r^2+(2-4r)\tilde{z}+r(r-1)\tilde{z}^2) = F_{1-r}(\tilde{z}). \end{aligned}$$

Therefore $g_{\alpha,\beta} \in \mathcal{K}S$ as well. Since $\alpha \neq 0$, we then have an example of a c-projective class of Kähler metrics with more affinely inequivalent CSC metrics.

Proposition 5.9. *On the stage* 3 *Bott manifold* $M_3(0,1,-1)$ *of twist* 1 *there exist an infinite number of pairs of c-projectively equivalent CSC Kähler metrics which are not affinely equivalent.*

Remark 5.10. At the moment it seems that the example above is rather special. From Remark 5.8 it is natural to speculate whether the same phenomenon occurs for the presumed second family of CSC admissible metrics on

$$M_{2m+1} = \mathbb{P}(\mathbb{1} \oplus \mathcal{O}(\overbrace{1,\dots,1}^{m},\overbrace{-1,\dots,-1}^{m}) \to \overbrace{\mathbb{CP}^1 \times \cdots \times \mathbb{CP}^1}^{2m}$$

for $m > 1$. Due to the non-explicit nature of the second family for $m > 1$ it seems rather intractable to check in general, but a computer aided calculation suggests that the phenomenon does not occur for $m = 2, m = 3$, and $m = 4$.

5.5. **Extremal almost Kähler metrics on certain stage** 3 **Bott manifolds.** Extremal metrics on almost Kähler manifolds have been discussed in [AD03, Don02, ACGT11] and studied in detail by Lejmi [Lej10]. We now consider constructions of smooth almost Kähler extremal metrics on certain stage 3 Bott manifolds. Essentially we use the generalized Calabi construction set-up from the beginning of Section 5, but we do not attempt to satisfy the integrability condition on $\mathbf{H}$. We refer to [ACGT11, Appendix A & Section 4.3] for definitions and justification of the adaption of the generalized Calabi construction.

Consider the case when the fiber is the product $\mathbb{CP}^1 \times \mathbb{CP}^1$. In this case V of Definition 5.1 is simply $\mathbb{CP}^1 \times \mathbb{CP}^1$ and with the identification of $\mathfrak{t}$ and $\mathfrak{t}^*$ with $\mathbb{R}^2$, the polytope is a rectangle. We use the notation $z = (z_1, z_2)$ and for simplicity, we assume that the polytope is the square $[0,1] \times [0,1]$ with normals

$$u_1 = \langle 1, 0\rangle, \quad u_2 = \langle -1, 0\rangle, \quad u_3 = \langle 0, -1\rangle, \quad u_4 = \langle 0, 1\rangle$$

on $F_1 = \{(0, z_2) \,|\, 0 \leq z_2 \leq 1\}$, $F_2 = \{(1, z_2) \,|\, 0 \leq z_2 \leq 1\}$, $F_3 = \{(z_1, 1) \,|\, 0 \leq z_1 \leq 1\}$, and $F_4 = \{(z_1, 0) \,|\, 0 \leq z_1 \leq 1\}$ respectively. An $S^2\mathfrak{t}^*$-valued function H on $\mathring{\Delta}$ may then be viewed as a matrix $\mathbf{H} = (H_{ij})$, where each entry H_{ij} is a smooth function of (z_1, z_2). Likewise $p \in \mathfrak{t}$ is just $p = (p_1, p_2)$, where here p_1 and p_2 must be integers. Note that p_1 and p_2 may be identified with a and b in the notation from Section 2.3. Of course c from Section 2.3 vanishes here. Now p_0 from Definition 5.1 must be a real constant such that $Q = Q(z_1, z_2) = p_0 + p_1 z_1 + p_2 z_2$ is positive for $(z_1, z_2) \in [0,1] \times [0,1]$. In particular, we need to assume that $p_0 \in \mathbb{R}^+$, $p_0 + p_1 > 0$, $p_0 + p_2 > 0$, and $p_0 + p_1 + p_2 > 0$.

The boundary conditions (56) on $\mathbf{H} = \begin{bmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{bmatrix}$ are

$$H_{11}(0, z_2) = H_{12}(0, z_2) = 0 \quad \text{and} \quad H_{11,1}(0, z_2) = 2 \quad \text{for} \quad 0 \leq z_2 \leq 1, \tag{64}$$

$$H_{11}(1, z_2) = H_{12}(1, z_2) = 0, \quad \text{and} \quad H_{11,1}(1, z_2) = -2 \quad \text{for} \quad 0 \leq z_2 \leq 1, \tag{65}$$

$$H_{22}(z_1, 1) = H_{12}(z_1, 1) = 0, \quad \text{and} \quad H_{22,2}(z_1, 1) = -2 \quad \text{for} \quad 0 \leq z_1 \leq 1, \tag{66}$$

$$H_{22}(z_1, 0) = H_{12}(z_1, 0) = 0, \quad \text{and} \quad H_{22,2}(z_1, 0) = 2 \quad \text{for} \quad 0 \leq z_1 \leq 1, \tag{67}$$

where $H_{ij,k} := \frac{\partial}{\partial z_k} H_{ij}$. To get a genuine positive definite metric out of this we also need that $\mathbf{H}$ is positive definite.

To complete the picture, observe that the integrability condition on J is as follows: if $\mathbf{H}^{-1} = (H^{ij})$, then we must have that

$$\frac{\partial H^{11}}{\partial z_2} = \frac{\partial H^{12}}{\partial z_1} \quad \text{and} \quad \frac{\partial H^{22}}{\partial z_1} = \frac{\partial H^{12}}{\partial z_2}. \tag{68}$$

The extremal condition is

$$4 - \frac{\partial^2}{\partial z_i \partial z_j}(QH_{ij}) = (A_1 z_1 + A_2 z_2 + A_3)Q$$

for some constants A_1, A_2, A_3 with CSC corresponding to $A_1 = A_2 = 0$.

Now assume that for any $i \leq j \in \{1, 2\}$, QH_{ij} is a polynomial P_{ij} of degree 4 in z_1, z_2. This gives the following equation for extremality of the corresponding compatible metric.

$$4 - P_{11,11} - 2P_{12,12} - P_{22,22} = (A_1 z_1 + A_2 z_2 + A_3)(p_0 + p_1 z_1 + p_2 z_2), \tag{69}$$

where $P_{ij,kl} := \frac{\partial^2 P_{ij}}{\partial z_k \partial z_l}$. Conditions (64)–(67) mean that for $0 \leq z_1 \leq 1$ and $0 \leq z_2 \leq 1$,

$$\begin{aligned}
&P_{11}(0, z_2) = P_{12}(0, z_2) = 0, && P_{11,1}(0, z_2) = 2Q(0, z_2) = 2(p_0 + p_2 z_2)\\
&P_{11}(1, z_2) = P_{12}(1, z_2) = 0, && P_{11,1}(1, z_2) = -2Q(1, z_2) = -2(p_0 + p_1 + p_2 z_2)\\
&P_{22}(z_1, 1) = P_{12}(z_1, 1) = 0, && P_{22,2}(z_1, 1) = -2Q(z_1, 1) = -2(p_0 + p_1 z_1 + p_2)\\
&P_{22}(z_1, 0) = P_{12}(z_1, 0) = 0, && P_{22,2}(z_1, 0) = 2Q(z_1, 0) = 2(p_0 + p_1 z_1).
\end{aligned}$$

These equations imply that

$$\begin{aligned}
P_{11} &= z_1(1 - z_1)(2Q + a_{11} z_1(1 - z_1))\\
P_{12} &= a_{12} z_1(1 - z_1) z_2(1 - z_2)\\
P_{22} &= z_2(1 - z_2)(2Q + a_{22} z_2(1 - z_2)),
\end{aligned} \tag{70}$$

for some constants $a_{ij} \in \mathbb{R}$.

Now we substitute these functions into the left hand side of (69) to get

$$\begin{aligned}
&(4 - 4p_1 - 4p_2 - 2a_{11} - 2a_{22} - 2a_{12} + 8p_0) + (12a_{11} + 4a_{12} + 16p_1)z_1\\
&\qquad + (12a_{22} + 4a_{12} + 16p_2)z_2 - 12a_{11}z_1^2 - 8a_{12}z_1 z_2 - 12a_{22}z_2^2.
\end{aligned}$$

Expanding the right hand side of (69) yields

$$p_0 A_3 + (p_0 A_1 + p_1 A_3)z_1 + (p_0 A_2 + p_2 A_3)z_2 + p_1 A_1 z_1^2 + (p_1 A_2 + p_2 A_1)z_1 z_2 + p_2 A_2 z_2^2.$$

It is then clear that - assuming p_0, p_1, p_2 fixed - we need to solve the linear system of 6 equations with 6 unknowns:

$$\begin{bmatrix}
-2 & -2 & -2 & 0 & 0 & -p_0\\
12 & 4 & 0 & -p_0 & 0 & -p_1\\
0 & 4 & 12 & 0 & -p_0 & -p_2\\
-12 & 0 & 0 & -p_1 & 0 & 0\\
0 & -8 & 0 & -p_2 & -p_1 & 0\\
0 & 0 & -12 & 0 & -p_2 & 0
\end{bmatrix}
\begin{bmatrix} a_{11}\\ a_{12}\\ a_{22}\\ A_1\\ A_2\\ A_3 \end{bmatrix}
=
\begin{bmatrix}
-4 + 4p_1 + 4p_2 - 8p_0\\
-16p_1\\
-16p_2\\
0\\
0\\
0
\end{bmatrix}.$$

This has a unique solution for $(a_{11}, a_{12}, a_{22}, A_1, A_2, A_3)$ since Q is positive over $[0, 1] \times [0, 1]$, so the determinant $6p_0^2 + 6p_0 p_1 + p_1^2 + 6p_0 p_2 + 3p_1 p_2 + p_2^2$ is positive.

We find (not unexpectedly) that $A_1 = A_2 = 0$ is only possible if $p_1 = p_2 = 0$ which yields the Kähler product $\mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$. In other words, none of the solutions yield non-trivial CSC almost Kähler metrics.

Since Q is assumed positive over $[0,1] \times [0,1]$, checking positive definiteness of $\mathbf{H}$ over $(0,1) \times (0,1)$ is equivalent to ensuring that $\mathbf{P}$ is positive definite over $(0,1) \times (0,1)$. The latter condition amounts to checking that

$$\det \mathbf{P} = P_{11}P_{22} - P_{12}^2 > 0 \quad \text{and} \quad P_{11} > 0$$

over $(0,1) \times (0,1)$. It is easy to see that for fixed p_1 and p_2 this is satisfied for sufficiently large and positive p_0 values. This is seen by the fact that the limit, $p_0 \to +\infty$ corresponds to having the product of Fubini–Study metrics on the fiber $\mathbb{CP}^1 \times \mathbb{CP}^1$. This is not unexpected in light of [ACGT11, Theorem 3].

To be more specific, observe now that $P_{11}P_{22} - P_{12}^2$ is

$$z_1(1-z_1)z_2(1-z_2)\big((2Q + a_{11}z_1(1-z_1))\,(2Q + a_{22}z_2(1-z_2)) - a_{12}^2 z_1(1-z_1)z_2(1-z_2)\big)$$

and recall that

$$P_{11} = z_1(1-z_1)(2Q + a_{11}z_1(1-z_1)).$$

Thus for positivity in a specific case we need

$$(2Q + a_{11}z_1(1-z_1))(2Q + a_{22}z_2(1-z_2)) - a_{12}^2 z_1(1-z_1)z_2(1-z_2) > 0, \tag{71}$$

$$(2Q + a_{11}z_1(1-z_1)) > 0. \tag{72}$$

Now assume also that p_1, p_2 are positive. In particular, each of them is at least one. It is not hard to check that the left hand side of (72) may be viewed as a second order polynomial in z_1 which is positive at $z_1 = 0$ and $z_1 = 1$ and has positive derivative at both these points as well. Thus (72) easily follows. The left hand side of (71) is a polynomial in p_0, p_1, p_2, z_1, and z_2. Viewing this as a polynomial of degree 6 in p_0, it is easy to check that the coefficients of $p_0^3, p_0^4, p_0^5, p_0^6$ are all positive. Using carefully that $p_1, p_2 \in \mathbb{Z}^+$ and $(z_1, z_2) \in (0,1) \times (0,1)$, it is also a relatively straightforward exercise to check that the coefficients of p_0^0 (the constant term w.r.t. p_0), p_0, and p_0^2 are positive.

We conclude that for $p_1, p_2 > 0$ we have an explicit almost Kähler extremal metric for each $p_0 \in \mathbb{R}^+$. Using computer aided algebra one may check that for these examples it is not possible to satisfy (68). Thus, the examples are non-Kähler, almost Kähler.

If we combine the constructions above, for p_1 and p_2 even, with the result in Proposition 2.9 we have the following result:

Proposition 5.11. *The smooth manifold $(\mathbb{CP}^1)^3$ admits uncountably many explicit non-CSC, non-Kähler, extremal almost Kähler metrics.*

Remark 5.12. Although the generalized Calabi and admissible constructions do not seem to lend themselves readily to producing new explicit examples of e.g. smooth stage 3 Bott manifolds with extremal (integrable) Kähler metrics, direct calculations suggest that it is possible to produce orbifold examples when the fiber is a non-trivial Hirzebruch surface. Indeed, one can combine the notion of Calabi toric Kähler metrics from [Leg11] with the generalized Calabi construction as introduced in [ACGT04], to construct explicit examples of admissible extremal Kähler metrics on such orbibundles whose fibers are Hirzebruch orbifolds.

References

[Abc13] Anna Abczynski, *On the classification of cohomology Bott manifolds*, Ph.D. thesis, Rheinischen Friedrich-Wilhelms-Universität, Bonn, 2013.

[ACG06] Vestislav Apostolov, David M. J. Calderbank, and Paul Gauduchon, *Hamiltonian 2-forms in Kähler geometry. I. General theory*, J. Differential Geom. **73** (2006), no. 3, 359–412. MR 2228318 (2007b:53149)

[ACGT04] Vestislav Apostolov, David M. J. Calderbank, Paul Gauduchon, and Christina W. Tønnesen-Friedman, *Hamiltonian 2-forms in Kähler geometry. II. Global classification*, J. Differential Geom. **68** (2004), no. 2, 277–345. MR 2144249

[ACGT08a] ———, *Hamiltonian 2-forms in Kähler geometry. III. Extremal metrics and stability*, Invent. Math. **173** (2008), no. 3, 547–601. MR MR2425136 (2009m:32043)

[ACGT08b] ———, *Hamiltonian 2-forms in Kähler geometry. IV. Weakly Bochner-flat Kähler manifolds*, Comm. Anal. Geom. **16** (2008), no. 1, 91–126. MR 2411469 (2010c:32043)

[ACGT11] ———, *Extremal Kähler metrics on projective bundles over a curve*, Adv. Math. **227** (2011), 2385–2424.

[AD03] Vestislav Apostolov and Tedi Draghici, *The curvature and the integrability of almost-Kähler manifolds: a survey*, Symplectic and contact topology: interactions and perspectives (Toronto, ON/Montreal, QC, 2001), Fields Inst. Commun., vol. 35, Amer. Math. Soc., Providence, RI, 2003, pp. 25–53. MR 1969266

[AGM93] Paul S. Aspinwall, Brian R. Greene, and David R. Morrison, *The monomial-divisor mirror map*, Internat. Math. Res. Notices (1993), no. 12, 319–337. MR 1253648 (95b:14029)

[Bat81] Victor V. Batyrev, *Toric Fano threefolds*, Izv. Akad. Nauk SSSR Ser. Mat. **45** (1981), no. 4, 704–717, 927. MR 631434

[Bat91] ———, *On the classification of smooth projective toric varieties*, Tohoku Math. J. (2) **43** (1991), no. 4, 569–585. MR 1133869

[Bat93] ———, *Quantum cohomology rings of toric manifolds*, Astérisque (1993), no. 218, 9–34, Journées de Géométrie Algébrique d'Orsay (Orsay, 1992). MR 1265307

[Bat99] ———, *On the classification of toric Fano 4-folds*, J. Math. Sci. (New York) **94** (1999), no. 1, 1021–1050, Algebraic geometry, 9. MR 2000e:14088

[BP15] Victor M. Buchstaber and Taras E. Panov, *Toric topology*, Mathematical Surveys and Monographs, vol. 204, American Mathematical Society, Providence, RI, 2015. MR 3363157

[BS58] Raoul Bott and Hans Samelson, *Applications of the theory of Morse to symmetric spaces*, Amer. J. Math. **80** (1958), 964–1029. MR 0105694 (21 #4430)

[BS99] Victor V. Batyrev and Elena N. Selivanova, *Einstein-Kähler metrics on symmetric toric Fano manifolds*, J. Reine Angew. Math. **512** (1999), 225–236. MR 2000j:32038

[Cal82] Eugenio Calabi, *Extremal Kähler metrics*, Seminar on Differential Geometry, Ann. of Math. Stud., vol. 102, Princeton Univ. Press, Princeton, N.J., 1982, pp. 259–290. MR 83i:53088

[CEMN16] David M. J. Calderbank, Michael G. Eastwood, Vladimir S. Matveev, and Katharina Neusser, *C-projective geometry*, arXiv:1512.04516v2 (2016), 117 pp.

[Cha17a] B. Narasimha Chary, *A note on toric degenerations of a Bott-Samelson-Demazure-Hansen variety*, preprint, arXiv: math.AG/1710.06300 (2017), 1–20.

[Cha17b] ———, *On Mori cone of Bott towers*, preprint, arXiv: math.AG/1706.02139 (2017), 1–47.

[Cho15] Suyoung Choi, *Classification of Bott manifolds up to dimension 8*, Proc. Edinb. Math. Soc. (2) **58** (2015), no. 3, 653–659. MR 3391366

[Civ05] Yusuf Civan, *Bott towers, crosspolytopes and torus actions*, Geom. Dedicata **113** (2005), 55–74. MR 2171298

[CLS11] David A. Cox, John B. Little, and Henry K. Schenck, *Toric varieties*, Graduate Studies in Mathematics, vol. 124, American Mathematical Society, Providence, RI, 2011. MR 2810322 (2012g:14094)

[CM12] Suyoung Choi and Mikiya Masuda, *Classification of $\mathbb{Q}$-trivial Bott manifolds*, J. Symplectic Geom. **10** (2012), no. 3, 447–461. MR 2983437

[CMM15] Suyoung Choi, Mikiya Masuda, and Satoshi Murai, *Invariance of Pontrjagin classes for Bott manifolds*, Algebr. Geom. Topol. **15** (2015), no. 2, 965–986. MR 3342682

[CMS10] Suyoung Choi, Mikiya Masuda, and Dong Youp Suh, *Topological classification of generalized Bott towers*, Trans. Amer. Math. Soc. **362** (2010), no. 2, 1097–1112. MR 2551516 (2011a:57050)

[CMS11] ———, *Rigidity problems in toric topology: a survey*, Tr. Mat. Inst. Steklova **275** (2011), no. Klassicheskaya i Sovremennaya Matematika v Pole Deyatelnosti Borisa Nikolaevicha Delone, 188–201. MR 2962979

[Cox97] David A. Cox, *Recent developments in toric geometry*, Algebraic geometry—Santa Cruz 1995, Proc. Sympos. Pure Math., vol. 62, Amer. Math. Soc., Providence, RI, 1997, pp. 389–436. MR 1492541 (99d:14054)

[CR05] Yusuf Civan and Nigel Ray, *Homotopy decompositions and K-theory of Bott towers*, K-Theory **34** (2005), no. 1, 1–33. MR 2162899 (2006g:55019)

[CS11] Suyoung Choi and Dong Youp Suh, *Properties of Bott manifolds and cohomological rigidity*, Algebr. Geom. Topol. **11** (2011), no. 2, 1053–1076. MR 2792373

[CvR09] David A. Cox and Christine von Renesse, *Primitive collections and toric varieties*, Tohoku Math. J. (2) **61** (2009), no. 3, 309–332. MR 2568257

[Dan78] Vladimir I. Danilov, *Geometry of toric varieties.*, Russ. Math. Surv. **33** (1978), no. 2, 97–154.

[Del88] Thomas Delzant, *Hamiltoniens périodiques et images convexes de l'application moment*, Bull. Soc. Math. France **116** (1988), no. 3, 315–339. MR 984900 (90b:58069)

[Dem70] Michel Demazure, *Sous-groupes algébriques de rang maximum du groupe de Cremona*, Ann. Sci. École Norm. Sup. (4) **3** (1970), 507–588. MR 0284446

[Don02] Simon K. Donaldson, *Scalar curvature and stability of toric varieties*, J. Differential Geom. **62** (2002), no. 2, 289–349. MR MR1988506 (2005c:32028)

[Don08] ———, *Kähler geometry on toric manifolds, and some other manifolds with large symmetry*, Handbook of geometric analysis. No. 1, Adv. Lect. Math. (ALM), vol. 7, Int. Press, Somerville, MA, 2008, pp. 29–75. MR 2483362 (2010h:32025)

[GK94] Michael Grossberg and Yael Karshon, *Bott towers, complete integrability, and the extended character of representations*, Duke Math. J. **76** (1994), no. 1, 23–58. MR 1301185 (96i:22030)

[Gro91] Michael Grossberg, *Complete integrability and geometrically induced representations*, Ph.D. thesis, Massachusetts Institute of Technology, 1991.

[Gua95] Daniel Guan, *Existence of extremal metrics on compact almost homogeneous Kähler manifolds with two ends*, Trans. Amer. Math. Soc. **347** (1995), no. 6, 2255–2262. MR 1285992 (96a:58059)

[Gui94] V. Guillemin, *Kaehler structures on toric varieties*, J. Differential Geom. **40** (1994), no. 2, 285–309. MR 1293656 (95h:32029)

[Hwa94] Andrew D. Hwang, *On existence of Kähler metrics with constant scalar curvature*, Osaka J. Math. **31** (1994), no. 3, 561–595. MR 1309403 (96a:53061)

[Ish12] Hiroaki Ishida, *Filtered cohomological rigidity of Bott towers*, Osaka J. Math. **49** (2012), no. 2, 515–522. MR 2945760

[Kar03] Yael Karshon, *Maximal tori in the symplectomorphism groups of Hirzebruch surfaces*, Math. Res. Lett. **10** (2003), no. 1, 125–132. MR 1960129 (2004f:53101)

[Koi90] Norihito Koiso, *On rotationally symmetric Hamilton's equation for Kähler-Einstein metrics*, Recent topics in differential and analytic geometry, Adv. Stud. Pure Math., vol. 18, Academic Press, Boston, MA, 1990, pp. 327–337. MR 1145263 (93d:53057)

[KS86] Norihito Koiso and Yusuke Sakane, *Nonhomogeneous Kähler-Einstein metrics on compact complex manifolds*, Curvature and topology of Riemannian manifolds (Katata, 1985), Lecture Notes in Math., vol. 1201, Springer, Berlin, 1986, pp. 165–179. MR 859583 (88c:53047)

[Leg11] Eveline Legendre, *Toric geometry of convex quadrilaterals*, J. Symplectic Geom. **9** (2011), no. 3, 343–385. MR 2817779 (2012g:53079)

[Lej10] Mehdi Lejmi, *Extremal almost-Kähler metrics*, Internat. J. Math. **21** (2010), no. 12, 1639–1662. MR 2747965

[Lic58] André Lichnerowicz, *Géométrie des groupes de transformations*, Travaux et Recherches Mathématiques, III. Dunod, Paris, 1958. MR 0124009

[LS93] Claude LeBrun and Santiago R. Simanca, *On the Kähler classes of extremal metrics*, Geometry and global analysis (Sendai, 1993), Tohoku Univ., Sendai, 1993, pp. 255–271. MR MR1361191 (96h:58037)

[LT97] Eugene Lerman and Susan Tolman, *Hamiltonian torus actions on symplectic orbifolds and toric varieties*, Trans. Amer. Math. Soc. **349** (1997), no. 10, 4201–4230. MR 98a:57043

[Mab87] Toshiki Mabuchi, *Einstein-Kähler forms, Futaki invariants and convex geometry on toric Fano varieties*, Osaka J. Math. **24** (1987), no. 4, 705–737. MR 89e:53074

[Mab14] ———, *Relative stability and extremal metrics*, J. Math. Soc. Japan **66** (2014), no. 2, 535–563. MR 3201825

[McD11] Dusa McDuff, *The topology of toric symplectic manifolds*, Geom. Topol. **15** (2011), no. 1, 145–190. MR 2776842 (2012e:53172)

[MP08] Mikiya Masuda and Taras E. Panov, *Semi-free circle actions, Bott towers, and quasitoric manifolds*, Mat. Sb. **199** (2008), no. 8, 95–122. MR 2452268 (2009i:57073)

[Nak93] Yasuhiro Nakagawa, *Einstein-Kähler toric Fano fourfolds*, Tohoku Math. J. (2) **45** (1993), no. 2, 297–310. MR 94f:32059

[Nak94] ———, *Classification of Einstein-Kähler toric Fano fourfolds*, Tohoku Math. J. (2) **46** (1994), no. 1, 125–133. MR 95e:32030

[Oda88] Tadao Oda, *Convex bodies and algebraic geometry*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 15, Springer-Verlag, Berlin, 1988, An introduction to the theory of toric varieties, Translated from the Japanese. MR 922894 (88m:14038)

[Sak86] Yusuke Sakane, *Examples of compact Einstein Kähler manifolds with positive Ricci tensor*, Osaka J. Math. **23** (1986), no. 3, 585–616. MR 866267 (87k:53111)

[SS11] Jacopo Stoppa and Gábor Székelyhidi, *Relative K-stability of extremal metrics*, J. Eur. Math. Soc. (JEMS) **13** (2011), no. 4, 899–909. MR 2800479

[WW82] Keiichi Watanabe and Masayuki Watanabe, *The classification of Fano* 3*-folds with torus embeddings*, Tokyo J. Math. **5** (1982), no. 1, 37–48. MR 670903

[WZ04] Xu-Jia Wang and Xiaohua Zhu, *Kähler-Ricci solitons on toric manifolds with positive first Chern class*, Adv. Math. **188** (2004), no. 1, 87–103. MR MR2084775 (2005d:53074)

[WZ11] Xu-jia Wang and Bin Zhou, *On the existence and nonexistence of extremal metrics on toric Kähler surfaces*, Adv. Math. **226** (2011), no. 5, 4429–4455. MR 2770455

[WZ14] ———, *K-stability and canonical metrics on toric manifolds*, Bull. Inst. Math. Acad. Sin. (N.S.) **9** (2014), no. 1, 85–110. MR 3234970

[ZZ08] Bin Zhou and Xiaohua Zhu, *Relative K-stability and modified K-energy on toric manifolds*, Adv. Math. **219** (2008), no. 4, 1327–1362. MR 2450612

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