Einstein Metrics on Spheres

CHARLES BOYER

University of New Mexico

Bilbao, País Vasco, July, 2012
History

- General Relativity.

(Einstein) Use Riemannian geometry with Lorentz signature as a theory of gravity. Reasoning: total amount of energy and momentum in the universe should equal the curvature of the universe.

Energy and momentum is represented by a symmetric 2-tensor $T_{\mu\nu}$. There are exactly two symmetric 2-tensors in the theory, the Ricci curvature, $R_{\mu\nu}$, and the (Lorentzian) metric itself $g_{\mu\nu}$. So Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} = 8\pi T_{\mu\nu}$$

$s$ scalar curvature.

Later add ‘cosmological constant’ $\Lambda g_{\mu\nu}$ to r.h.s. “The biggest blunder of my life.” (Einstein)

But recently ‘not so big a blunder’—

dark energy $\Rightarrow \Lambda$ small but $> 0$. 

Riemannian manifold \((M, g)\)
A Riemannian metric \(g\) is Einstein if \(\text{Ric}_g = \lambda g\)
\(\lambda\) constant
Three cases:
(1) \(\lambda > 0\),
(2) \(\lambda = 0\),
(3) \(\lambda < 0\).

Motivation

● Variational Principle
Normalize: (vol of \(g\)) = 1.
\[ g \mapsto \int_M s_g d\mu_g, \mu_g \text{ volume} \]
(Hilbert) Einstein metrics are critical points.

Quadratic functionals:
\[ g \mapsto \int_M s_g^2 d\mu_g \text{ (Calabi)} \]
Einstein metrics are critical points. Maybe Einstein metrics are distinguished.
Spheres

- **History**: Einstein metrics
- Round metric on $S^n$ (Gauss-Riemann)
- Squashed metrics on $S^{4n+3}$ (Jensen, 1973)
- Homogeneous Einstein metric on $S^{15}$ (Bourguignon and Karcher, 1978).
- These are all homogeneous Einstein metrics on $S^n$ and they are the only such metrics up to homothety (Ziller, 1982).
- Infinite sequences of inhomogeneous Einstein metrics on $S^5, S^6, S^7, S^8$ and $S^9$ (Böhm, 1998). Maybe not so distinguished
Exotic Spheres

(Milnor, 1956)
Spheres that are homeomorphic but not diffeomorphic to $S^n$. Homotopy spheres that bound a parallelizable manifold $bP_{n+1}$ form an Abelian group. (Kervaire-Milnor)
For $S^{4n+1}$, $bP_{4n+2} = \mathbb{Z}_2$ if $4n \neq 2^j - 4$ for any $j$.
No $bP$ exotics $S^5, S^{13}, S^{29}, S^{61}$.

$bP_8 = \mathbb{Z}_{28}, bP_{12} = \mathbb{Z}_{992}$
$bP_{16} = \mathbb{Z}_{8128}, bP_{20} = \mathbb{Z}_{130816}$
Generally, $|bP_{4m}| = 2^{2m-2}(2^{2m-1} - 1) \text{ num } (\frac{4B_m}{m})$

• Results (B-, Galicki, Kollár)
$N_{SE} = \#$ of deformation classes Einstein metrics.
$\mu_{SE} = \#$ moduli of Einstein metrics.
• Each 28 diffeo types of $S^7$ admits hundreds of Einstein metrics, many with moduli. Largest moduli has dimension 82, standard $S^7$.

• All 992 diffeo types in $bP_{12}$ and all 8128 diffeo types in $bP_{16}$ admit Einstein metrics, i.e. on $S^{11}, S^{15}$.
• All elements of $bP_{4n+2}$ admit Einstein metrics.

Our Einstein metrics are special, Sasaki-Einstein (SE)

• Both the number $N_{SE}$ of deformation classes and the number $\mu_{SE}$ of moduli grow double exponentially with dimension.

(1) $N_{SE}(S^{13}) > 10^9$ and
$\mu_{SE}(S^{13}) = 21300113901610$

(2) $N_{SE}(S^{29}) > 5 \times 10^{1666}$ and
$\mu_{SE}(S^{29}) > 2 \times 10^{1667}$

**Conjecture:** Both $N_{SE}(S^{2n-1})$ and $\mu_{SE}(S^{2n-1})$ are finite.
Similar results for rational homology spheres $(\mathcal{B}, \text{Galicki})$ and other manifolds.

**Ingredients of Proof**

1. **Contact geometry.** Sasakian metrics
2. **Differential topology.** Diffeomorphism types
3. **Singularity theory.** Links of isolated hyper-surface singularities
4. **Algebraic geometry.** Algebraic orbifolds
5. **Analysis.** Monge-Ampère deformations
Contact Manifold (compact)

A contact 1-form \( \eta \) such that

\[
\eta \wedge (d\eta)^n \neq 0.
\]

defines a contact structure

\[
\eta' \sim \eta \iff \eta' = f\eta
\]

for some \( f \neq 0 \), take \( f > 0 \). or equivalently a codimension 1 subbundle \( D = \text{Ker} \, \eta \) of \( TM \).

\((D, d\eta)\) symplectic vector bundle

Unique vector field \( \xi \), called the Reeb vector field, satisfying

\[
\xi|_{\eta} = 1, \quad \xi|d\eta = 0.
\]

The characteristic foliation \( \mathcal{F}_\xi \) each leaf of \( \mathcal{F}_\xi \) passes through any nbd \( U \) at most \( k \) times \( \iff \) quasi-regular, \( k = 1 \) \( \iff \) regular, otherwise irregular
Contact bundle $\mathcal{D} \rightarrow$ choose **almost complex structure** $J$ extend to $\Phi$ with $\Phi \xi = 0$

Get a compatible metric

$$g = d\eta \circ (\Phi \otimes 1 + \eta \otimes \eta)$$

Quadruple $S = (\xi, \eta, \Phi, g)$ called **contact metric structure**

**Definition:** The structure $S = (\xi, \eta, \Phi, g)$ is **K-contact** if $\mathcal{L}_\xi g = 0$ (or $\mathcal{L}_\xi \Phi = 0$). It is **Sasakian** if in addition $(\mathcal{D}, J)$ is integrable. 

**Note:** Here we work entirely with Sasakian structures.
Geometry of Links

$\mathbb{C}^{n+1}$ coord’s $z = (z_0, \ldots, z_n)$
weighted $\mathbb{C}^*$-action

$$(z_0, \ldots, z_n) \mapsto (\lambda^{w_0}z_0, \ldots, \lambda^{w_n}z_n),$$

weight vector $w = (w_1, \ldots, w_n)$ with $w_j \in \mathbb{Z}^+$ and
$\gcd(w_0, \ldots, w_n) = 1$.

$f$ weighted homogeneous polynomial

$$f(\lambda^{w_0}z_0, \ldots, \lambda^{w_n}z_n) = \lambda^df(z_0, \ldots, z_n)$$

$d \in \mathbb{Z}^+$ is degree of $f$.

$0 \in \mathbb{C}^{n+1}$ isolated singularity.

**link** $L_f$ defined by

$$L_f = f^{-1}(0) \cap S^{2n+1},$$

$S^{2n+1}$ unit sphere in $\mathbb{C}^{n+1}$

Special Case: Brieskorn-Pham poly. (BP)

$$f(z_0, \ldots, z_n) = z_0^{a_0} + \cdots + z_n^{a_n}$$

$a_iw_i = d, \ \forall i$. 
Brieskorn-Pham Graph Thm:

For \( \mathbf{a} = (a_0, \ldots, a_n) \) integers \( \geq 2 \Rightarrow \) a graph \( G(\mathbf{a}) \) whose vertices are \( a_i \). And \( a_i \) is connected to \( a_j \) if \( \text{gcd}(a_i, a_j) > 1 \).

Link \( L_f \) is a homology sphere \( \iff \)
(1): \( G(\mathbf{a}) \) contains at least two isolated points, or
(2): \( G(\mathbf{a}) \) has an odd \# of vertices and \( a_i, a_j \), \( \text{gcd}(a_i, a_j) = 2 \) if \( \text{gcd}(a_i, a_j) > 1 \).

Determine the diffeomorphism type:
(1): If \( \text{dim} \equiv 3 \text{mod } 4 \): given by Hirzebruch signature of manifold that \( L_f \) bounds. Combinatorial formula (Brieskorn)
(2): If \( \text{dim} \equiv 1 \text{mod } 4 \): \( G(\mathbf{a}) \) has one isolated point \( a_k \) such that \( a_k \equiv \pm 3 \text{mod } 8 \) gives KerVAIRE sphere. \( a_k \equiv \pm 1 \text{mod } 8 \) gives standard sphere.
Fact: $L_f$ has natural structure with commutative diagram: $S^2_{w}^{2n+1}$ weight sphere
$\mathbb{P}_{\mathbb{C}}(w)$ weighted projective space

$$
\begin{align*}
L_f & \longrightarrow S^2_{w}^{2n+1} \\
\downarrow \pi & \downarrow \\
\mathcal{Z}_f & \longrightarrow \mathbb{P}_{\mathbb{C}}(w),
\end{align*}
$$

horizontal arrows: Sasakian and Kählerian embeddings.
vertical arrows: orbifold Riemannian submersions.

$L_f$ is Sasaki-Einstein (SE) $\iff$ $\mathcal{Z}_f$ is Kähler-Einstein (KE)

**Question**: When do we have SE or KE metrics?
1. $c_{1}^{orb}(\mathcal{Z}) > 0$ (easy)
2. solve Monge-Ampère equation (hard)

$$
\frac{\det(g_{ij}^{-} + \frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j})}{\det(g_{ij}^{-})} = e^{f-t\phi}.
$$
Tian: uniform boundedness

\[
\int_Z e^{-\gamma t \phi_0} \omega_0^n < +\infty
\]

Many people Yau, Tian, Siu, Nadel, and most recently by Demailly and Kollár in orbifold category.

**algebraic geometry of orbifolds:**
local uniformizing covers

**branch divisor:** \( \mathbb{Q} \)-divisor

\[
\Delta := \sum (1 - \frac{1}{m_j}) D_j
\]

**canonical orbibundle**

\[
K_{\text{orb}}^Z = K_Z + \sum (1 - \frac{1}{m_j}) [D_j],
\]

ramification index: \( m_j \)
Kawamata log terminal or klt For every \( s \geq 1 \)
and holomorphic section \( \tau_s \in H^0(Z, \mathcal{O}(\text{K}_{\text{orb}}^Z)^{-s}) \)
there is \( \gamma > \frac{n}{n+1} \) such that \( |\tau_s|^{-\gamma/s} \in L^2(Z) \).
Theorem 2: $c_1^{\text{orb}}(\mathcal{Z}) > 0$, klt $\Rightarrow$ Sasaki-Einstein metric.

**Sasaki-Einstein metrics**

Positivity $\Rightarrow I = (\sum w_i - d) > 0$

klt estimates for $L_f$

$$d(\sum w_i - d) < \frac{n}{n-1} \min_{i,j} w_i w_j.$$

BP polyn: (better)

$$1 < \sum_{i=0}^n \frac{1}{a_i} < 1 + \frac{n}{n-1} \min_i \left\{ \frac{1}{a_i}, \frac{1}{b_i b_j} \right\}.$$  

$a_i$ BP exponents and

$$b_i = \gcd(a_i, \text{lcm}(a_j | j \neq i)).$$

$\exists$ other estimates. Positivity plus a klt estimate $\Rightarrow$ SE metric

To determine the moduli $\mu_{SE}$ add monomials $z_{i_1}^{b_{i_1}} \cdots z_{i_k}^{b_{i_k}}$ such that $\sum_j b_{i,j} = d$ to BP polynomial. Divide by equivalence of $\text{Aut}(\mathcal{Z}_f)$.
Why double exponential growth?

Reason for growth: Sylvester’s sequence determined by $c_{k+1} = 1+c_0 \cdots c_k$ begins as $2, 3, 7, 43, 1807, 3263443, 10650056950807, \ldots$

$N_{SE}$: sequences $a = (a_0 = c_0, \ldots, a_{n-1} = c_{n-1}, a_n)$ with $c_{n-1} < a_n < c_0 \cdots c_{n-1}$ give SE metrics. Use prime number theorem.

$\mu_{SE}$: sequences $a = (a_0 = c_0, \ldots, a_{n-1} = c_{n-1}, a_n)$ where $a_n = (c_{n-1} - 2)c_{n-1}$. Polynomial $f$ contains $G(z_{n-1}, z_{c_{n-1}^{-2}})$. Again by prime number theorem gives double exponential growth.

**Conjecture:** All elements of $bP_{2n}$ admit SE metrics.

Estimate of Lichnerowicz $\Rightarrow$ if $I = (\sum w_i - d) > n \min_i w_i$ then $\not\exists$ SE metrics. Only applies to KE orbifolds! (Gauntlett,Martelli,Sparks,Yau) (Ghigi,Kollár) class of SE metrics where bound is sharp. $\times 10$ more on spheres.