Hilbert, Turing, Matijasevich

Data

A seq. $\{T_1, T_2, T_3, \ldots\}$ of $T_n : N \to N \cup \{\text{STOP}\}$ (Assume some output is $\text{STOP}$)

$T_n$ (called $n$-th Turing machine); $m \in N$ put $T_n(m)$ called $A$ output if $T_n(m) = \text{STOP}$, say "computation doesn't stop for input $m$ & machine $T_n$".

A sequence $\{a_1, a_2, \ldots\}$ in $N$ is called computable if $\exists \; \text{an } \text{m s.t. } T_n(c) = a_c$.

A set $S \subset N$ is listable if $\exists \; T_n \text{ s.t. } \text{Im } T_n = S$

A set $S \subset N$ is decidable if $\exists \; \text{m s.t. } \text{Im } T_n = \{0, 1\}^* \& \forall x \in N, T_n(x) = 1 \forall x \in S$

$S$ decidable $\iff$ $S \& N \setminus S$ listable

Proposition 1. Let $T_p \& T_q$ give $S \& N \setminus S$, make them "run alternatively" i.e. consider $T_p(1), T_q(1), T_p(2), T_q(2), \ldots$. If $x$ shows up in every place define $T(x) = 1$ if repeated.

By (A) this $T$ must be a $T_r$.

" $\Rightarrow$ " Say $T_n$ "decides" $S$, consider $T_n(1), T_n(2), \ldots$; $T_n(c) = 4$ if $c \in S$, put it in $N \setminus S$. Take $N^2 \subset N$ given by $\{ (1, 1), (1, 2), (2, 1), (2, 2), (3, 1), (3, 2), \ldots \}$ & view $T_n : N \times N \to N \cup \{\text{STOP}\}$

(Turing) The set $S \subset N \times N, S = \{ (m, n) \mid T_n(m) \neq \text{STOP} \}$ is listable (computable) & non-deciable.

Proposition 2. Let $T_1, T_2, T_3, \ldots$ run w/ inputs $(1, 2, 3)$ as follows:

As soon as a comp stops list $(m, n)$.

So $S$ enumerable. Assume decidable.

Then $\exists$ machine that can compute table $m \mapsto T_n(m)$ (one digit at a time, one digit at a time)

Now replace all $\infty$ by 0 & add 1 on diag. This table is compatible (while doing that), so the diagonal is computable so the diag must be one of the rows of $T$, (which is not &).

$$\forall m \exists T_n \; : \; N \to \{0, 1, 2, \ldots\} \& \forall m \in N \cup \{\text{STOP}\} \forall x \in N, T_n(m) = s_x$$

or enumerated

$$\text{Internal states} \; \frac{T_n(m) = (s_{m_1}, s_{m_2}, \ldots)}{\text{STOP} \ldots}$$
\[ S \subseteq \mathbb{N} \text{ called Diophantine if } \exists \text{ polynomials in } n \text{ vars } \exists \alpha, \beta \in \mathbb{N} \text{ s.t. } \alpha, \beta \in S \Rightarrow F(x, y) = 0 \text{ has solution in } \mathbb{Z}^n. \]

\[ S \text{ Diophantine } \Rightarrow S \text{ listable } \]

\[ \exists \text{ Dioph } S \subseteq \mathbb{N} \text{ which is not decidable.} \]

\[ \text{Matijasevich} \] \[ S \text{ listable } \Rightarrow S \text{ Diophantine.} \]