HISTORY OF GALOIS THEORY

Part I: Before Galois

1. Babylonians (2000 BC)
   Greeks (Euclid 300 BC)
   Arabs (Omar Khayyam 1000 AD)

   \( x^2 + bx + c = 0 \)
   \( x = \frac{-b \pm \sqrt{b^2 - 4c}}{2} \)
   \( \Delta = b^2 - 4c \)

2. Plato 300's BC:
   - Can one construct \( \sqrt{2} \) with straightedge & compass?
   - (Equivalently: Can one solve \( x^3 - 2 = 0 \) using only sq roots?
   - (Similarly: Can one construct reg poly's, trisect angles, etc.)

3. del Ferro 1500's
   Tartaglia (won contest)
   Cardano (disclosed)

   \( x^3 + px + q = 0 \)
   \( u^3 + v^3 = -q \)
   \( 3uv = -p \)

   Same for \( x^4 + ax^3 + bx^2 + cx + d = 0 \)

   Comment: Birth of \( \mathbb{C} \) (blc can happen \( \Delta < 0 \) \( x \in \mathbb{R} \))

4. Gauss 1800's

   Th (FTA) Any \( f \in \mathbb{C}[X] \) is a prod of lin factors (so has roots in \( \mathbb{C} \)).
   Th \( x^m - 1 = 0 \) is "solvable by radicals of deg < m"
   Th \( x^p - 1 = 0 \) is "sol by rad's of deg 2 (square roots)"
   iff \( p = 2^m + 1 \) for some \( m \) (a Fermat prime)
   (equivalently, "a reg poly w/ p sides is constructible"
   iff \( p \) is Fermat).
   Th (indep: Wantzel) \( \sqrt{2} \), \( \cos 20^\circ \) not constructible (Plato's prob).

5. Abel 1800's

   Th \( x^5 + ax^4 + bx^3 + \ldots + e = 0 \) (where a,b,..., e indet) not
   "sol by rad's". (§)

\[ \ast \] for some \( p \) prime.
PART II  GALOIS & AFTER.

Galois 1800's

Th Let $f \in \mathbb{Q}[x]$ be irreducible (coeff #'s not letters!) & $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ its roots.
Let $K$ be the smallest subfield of $\mathbb{C}$ s.t. $\alpha_1, \ldots, \alpha_n \in K$. (note $K = \mathbb{Q}(\alpha_1, \ldots, \alpha_n)$)
Let $G = \text{Gal}(K/\mathbb{Q}) = \{ \sigma : K \to K \text{ field aut.} \}$ (finitegrp.)
Then $f = 0$ "sol by rad's" $\iff G$ solvable
(or: Gauss, Abel: same for $Q$ replaced by any subfield $L$ of $C$)
(Pbm 1: how to compute $G$ from $f$ w/o knowing $\alpha_i$'s OPEN
2: what $G$'s can occur? (Inverse Gal. problem) \{in general\}

Kronecker 1800's

Th Let $f$ as above be s.t. $G$ abelian. Then its roots $\alpha_i$ are all $\mathbb{Q}$-lin combinations of #s of the form $\zeta_n^i = \cos \frac{2k\pi i}{n} + i \sin \frac{2k\pi i}{n}$ (Rad.)
(note $\zeta_n^i = f(k/n)$, $f(z) = e^{2\pi iz}$)

$K$'s j纳斯 trauma. Prove synth similar w/ $\mathbb{Q}$ replaced by othersubfields

Hilbert: some of $K$'s “j纳斯” abelian $f(z) = e^{2\pi iz}$ repl by other boldness

Artin 1920's finished “class field th” program (exploit define $G(K/k)$ etc.)
Also w/ $f$ as above he attached to each prime $p \in \mathbb{Z}$
an elt $\sigma_p \in G$, he embedded $G \subset \text{GL}_n(\mathbb{C})$ & set $a_p = \text{tr}(\sigma_p)$ & constructed holo fn $L(z)$ from $a_p$'s.
He introduced the philosophy that $L$ should have some
c (hidden symmetry).
(Hidden symmetry.)

Shafarevich 1970

Th Any solvable group appears for some $f \in \mathbb{Q}[x]$.

Langlands ~1970, .... Wiles 1995

Cases when Artin's $L$ has the beautiful props.
& alg. geo. analogues Non-abel class field th ????