Given \( M = \mathbb{R}^n \) (space or space time)

A Lie group acting on \( M \) (hence on \( L^2(M) \)) and on a finite dim vect \( \mathcal{V} \)

\( \mathcal{G} = H \rightarrow H \) a \( G \)-equiv linear oper (densely defined !)

\[ \mathcal{L} = -\frac{i}{\hbar} \frac{\partial}{\partial t} + \mathcal{V} \]

Examples

1) \( M = \mathbb{R}^3 \), \( G = SO(3) \), \( \mathcal{V} = \mathbb{R}^3 \), \( \mathcal{L} = -\frac{\hbar^2}{2\mu} \Delta + \mathcal{V} \), \( \mathcal{V} = V(r) \).

2) \( M = \mathbb{R}^4 \), \( G = \text{Lorentz} \), \( \mathcal{V} = \mathbb{C}^4 \) w/ \( G \) acting via 

\[ (A, \mathcal{A}^\dagger) \text{ where } A \in SL(2, \mathbb{C}) \]

\( \mathcal{L} = \text{Dirac equiv op.} \)

Back to general sit. Let \( R = (\text{clos of }) \ker \mathcal{L} \), Then \( G \) acts on \( R \)

Define a Casimir oper as an elt of the center of \( \mathcal{U}(G) \) where \( g = \text{Lie } G \).

If \( W \) is an irreducible rep of \( G \) then any Casimir oper \( C \) induces a 

G-act of \( W \) so is a scalar \( g \cdot Id_W \), \( g \in G \); call \( g \) the quantum num of \( W \) corre.

Define an eltary particle as an irr rep of \( G \)

Often \( R = \bigoplus W_j \), \( W_j \) irr. One interprets this by saying that a sol of \( \text{H} \)

is a superpos of eltary particles each of which has some definite quantum nums.

Say \( W_1, W_2 \) are irr rep's \( W \bigoplus W_j \approx \bigoplus V_k \), \( V_k \) irr. One interprets this by saying that a state of a pair of particles \( W_1 \& W_2 \) is a superpos of particles \( V_k \).

\( S^n(W_j) \) represents \( n \) bosons of type \( W_j \).

\( \bigoplus W_j \) in Fermions of type \( W_j \).

Example

\[ G = SO(3) = \{ A \in GL(3, \mathbb{R}) \mid \det A = 1 \}, AA^T = I \]

\( A = \frac{1}{2} \mathbb{1} \mid \det(I + EM) = 1 \), \( (I + EM)(I + EM^T) = I \)

\[ = \frac{1}{2} \mathbb{1} \mid trM = 0, M + M^T = 0 \] \( = \text{Span} \{ (1,0,0), (0,1,0), (0,0,1) \} \)

I + EM acts on \( C^\infty \text{ fun } f(x, y, z) \), \( f: \mathbb{R}^3 \rightarrow \mathbb{C} \) as follows

\[ ((I + EM) f)(x, y, z) = f(x + y, -x + y, 2) = f(-1 + 0, 0, 0) \]

\( \text{by Shur's Lemma} \)

Sometimes it is true that if \( W_1, W_2 \) have the same quantum nums \( \{ \text{i.e. } \mathcal{C}, q_c(W_1) = q_c(W_2) \} \)

then \( W_1 \& W_2 \) as \( G \)-mod \( 's \).
A Casimir operator (ess the only one) is \( C = x^2 + y^2 + z^2 \in U(3) \).

Action of \( C \) on \( C^0(\mathbb{R}^3) \) is by \( L^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \).

In quantum mechanics this op \( L \) is called "total angular momentum" \( \mathbf{L} \).

**L** Let \( \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \) & \( P_n = \text{four poles in } x, y, z \) of deg \( n \) w/ C-coeffs. Then

1. \( \Delta \) is \( SO(3) \)-equiv so \( SO(3) \) acts on \( \text{Ker} (\Delta: P_n \to P_{n-2}) = V_n \)
2. \( V_n \) is an irr rep of \( SO(3) \) and every irr rep of \( SO(3) \) is \( \cong \) some \( V_n \).

**P** The quantum number \( n \) \((V_n) = -n(n+1)\).

\[
\Phi_n(x+i y)^m \in V_n \text{ b/c } \Delta \Phi_n = (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}) \Phi_n = (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})(x+i y)^m = m(m+1) \Phi_n \text{ where } \Phi_n = \Phi_n(x+i y) \in \text{Ker} \Delta.
\]

Now \( (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}) \Phi_n = -m(m+1) \Phi_n \).

So \( \Phi_n(x+i y)^m \in V_n \).

Similarly, one computes \( \Phi_n(x+i y)^m \in \text{Ker} \Delta \).

We get \( L^2 \Phi_n = n(n+1) \Phi_n \).

**R** \( V_n \) appears as many times as \( L^2 \). Indeed \( \forall \Phi_n \in L^2(\mathbb{R}^3), \frac{\partial^2}{\partial x^2} \Phi_n = \Phi_n \).

**L** For \( \Phi \in C^0(\mathbb{R}^3) \), \( \Delta \Phi = (L^2) \Phi \) \text{ where } \Phi = \Phi(x+i y) \in V_n \).

**B** Direct comp (hope didn't make a mistake) (Use \( \Delta \Phi = 0 \), \( \Phi(x+i y) \in V_n \).

**D** Let \( \Phi_n \in V_n \) be a basis of \( V_n \).

**L** \( \Phi_n \in C^0(\mathbb{R}^3) \) is dual \( L^2(\mathbb{R}^3) \).

Using the above, one can study spectrum of \( L = -\frac{\hbar^2}{2m} \Delta + V \), \( V = V(r) \) as follows.

**R** Want to study \( \Phi \in L^2 \) \text{ s.t. } \Delta \Phi = E \Phi \text{. Assume } \Phi = \sum \Phi_n \Phi_n \text{ for } \Phi_n \in V_n \).

Eventually leads to \( V = \sum \Phi_n \Phi_n \).