Strichartz estimates in polygonal domains and cones

Matthew D. Blair

University of New Mexico

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Joint work with:
- G. Austin Ford (Northwestern)
- Jeremy Marzuola (North Carolina)
The wave equation on $\mathbb{R}^n$

- Initial value problem for the wave equation

$$\square u := (D^2_t - \Delta)u = 0, \quad (u, \partial_t u)|_{t=0} = (f, g),$$

$$u(t, x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}, \quad (D_t = -i\partial_t, \quad \Delta \geq 0)$$

- Properties:

$$\| \nabla_{t,x} u(t, \cdot) \|^2_{L^2} = \| \nabla_{t,x} u(0, \cdot) \|^2_{L^2} \quad \text{(energy conservation)}$$

$$\| u(t, \cdot) \|_{L^\infty(\mathbb{R}^n)} \leq C(u)(1 + |t|)^{-\frac{n-1}{2}} \quad \text{(decay inequality)}$$
Nonlinear wave equations

- Semilinear wave equation with power type nonlinearity
  \[ \Box u = \pm |u|^{r-1}u \]

- Inhomogeneous energy estimates
  \[ \| \nabla_{t,x} u(t, \cdot) \|_{L^2} \lesssim \| \nabla_{t,x} u(0, \cdot) \|_{L^2} + \int_0^t \| \Box u(s, \cdot) \|_{L^2} \, ds \]

- In order to linearize the equation, need to estimate powers of solutions efficiently
  \[ \| u^r \|_{L^1(I;L^2(\mathbb{R}^n))} = \| u \|_{r,L^r(I;L^{2r}(\mathbb{R}^n))}^r, \quad I = (-T, T) \]
Strichartz estimates

- Robert Strichartz (1970’s)–estimates for $\Box u = 0$:
  \[
  \|u\|_{L^q(\mathbb{R}^{n+1})} \leq C \left( \|f\|_{\dot{H}^1_x(\mathbb{R}^n)} + \|g\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{R}^n)} \right), \quad q = \frac{2(n+1)}{n-1}
  \]

- Consequence of Stein-Tomas restriction theorem: $\hat{u}(\tau, \xi)$ is supported on the cone $S = \{ \tau^2 = |\xi|^2 \}$,
  \[
  \|u\|_{L^q(\mathbb{R}^{n+1})} \leq C \|\hat{u}\|_{L^2(S)}
  \]
  which is dual to a Fourier restriction estimate
Strichartz estimates on $\mathbb{R}^n$
Boundary value problems
Estimates on cones

Strichartz estimates

- 80’s/90’s: Ginibre-Velo, Lindblad-Sogge, Keel-Tao, others

$$\|u\|_{L^p(\mathbb{R}; L^q(\mathbb{R}^n))} \leq C \left( \|f\|_{H^\gamma(\mathbb{R}^n)} + \|g\|_{H^{\gamma-1}(\mathbb{R}^n)} \right)$$

- Admissibility conditions:

1. $$\frac{1}{p} + \frac{n}{q} = \frac{n}{2} - \gamma \quad \text{(Scaling)}$$

2. $$\frac{2}{p} + \frac{n-1}{q} \leq \frac{n-1}{2} \quad \text{(Knapp example/Lorentz)}$$
Littlewood-Paley decompositions

- Take a Littlewood-Paley decomposition in the spatial frequencies

\[ u = \sum_{k=\infty}^{\infty} u_k, \quad u_k(t, \cdot) = \mathcal{F}^{-1}\{\beta_k(\xi)\hat{u}(t, \xi)\}, \]

\[ \text{supp}(\beta_k) \subset \left\{ 2^{k-\frac{1}{2}} < |\xi| < 2^{k+\frac{3}{2}} \right\}, \quad \sum_{k=\infty}^{\infty} \beta_k(\xi) = 1 \]

- The Littlewood-Paley squarefunction estimate reduces matters to

\[ \|u_k\|_{L^p(L^q)} \lesssim 2^{\gamma k} \|f_k\|_{L^2} + 2^{\gamma(k-1)} \|g_k\|_{L^2} \quad k \in \mathbb{Z} \]

- Use scale invariance \((t, x) \mapsto (2^{-k} t, 2^{-k} x)\) to reduce to

\[ \|u_0\|_{L^p(L^q)} \lesssim \|f_0\|_{L^2} + \|g_0\|_{L^2} \quad (k=0) \]
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Frequency localized estimates

- Crucial matter: show that
  \[ \| u_0(t, \cdot) \|_{L^\infty} \lesssim (1 + |t|)^{-\frac{n-1}{2}} \left( \| f_0 \|_{L^1} + \| g_0 \|_{L^1} \right) \]

- Oscillatory integral approach is most effective
  \[ \left| \int e^{i(x - y) \cdot \xi \pm it |\xi|} \alpha(|\xi|) \, d\xi \right| \lesssim (1 + |t|)^{-\frac{n-1}{2}}, \quad \alpha \in C^\infty_c(\mathbb{R}^n) \]

- Can view the Littlewood-Paley multiplier as an operator which regularizes the Schwartz (distributional) kernels of
  \[ \frac{\sin(t \sqrt{\Delta})}{\sqrt{\Delta}} \quad \text{and} \quad \cos(t \sqrt{\Delta}) \]
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Boundary value problems

Let $\Omega$ be a domain in $\mathbb{R}^n$, and consider solutions to

$$ (D_t^2 - \Delta)u = 0, \quad (u, \partial_t u)\big|_{t=0} = (f, g), $$

$$ u(t, \cdot)|_{\partial \Omega} = 0 \text{ (Dirichlet)} \quad \text{ or } \quad \frac{\partial u}{\partial \nu}(t, \cdot)|_{\partial \Omega} = 0 \text{ (Neumann)} $$

Boundary conditions affect the flow of energy

Trapped rays can preclude a global (in time) estimate
Boundary value problems

- Partial progress on smooth boundaries: Smith-Sogge, Burq-Lebeau-Planchon, MDB-Smith-Sogge

- Common thread—can construct a parametrix for the equation

- Domains with corners? No known effective parametrix
  - Melrose-Vasy-Wunsch: If a singularity lies on a ray which approaches a corner, it lies within the union of a family of rays after the interaction
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Sommerfeld’s example

Sommerfeld (1896) did explicit computations in the exterior of a wedge—he showed that when a wavefront interacts with the tip, a spherical wave of singularities is formed, even into the shadow region.

(Figure from Friedlander’s *Sound Pulses*)
Main theorem for domains

Theorem (MDB, Ford, Marzuola)

Let $\Omega$ be a domain in $\mathbb{R}^2$ whose boundary consists of a finite number of line segments. Then any solution to the wave equation with Dirichlet or Neumann BC’s satisfies

$$\|u\|_{L^p((-T,T);L^q(\Omega))} \lesssim \|f\|_{H^\gamma(\Omega)} + \|g\|_{H^{\gamma-1}(\Omega)}$$

$$\frac{1}{p} + \frac{2}{q} = 1 - \gamma \quad \text{(scaling)}$$

$$\frac{2}{p} + \frac{1}{q} \leq \frac{1}{2} \quad \text{(Knapp admissibility)}$$
Doubling the domain

- Since the estimate is local in time, finite speed of propagation means that it suffices to work locally in space, that is, over sets as small as you like.
- Away from the vertices: use the method of images.
- Near the vertices: impose polar coordinates \((r, \theta)\) centered at the vertex. If the angle is \(\alpha\), \((0, \delta) \times [0, \alpha] \subset \mathbb{R}_+ \times S^1\) will describe the neighborhood.
- This neighborhood can be “doubled” by gluing a copy of the corner on to the original.
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This neighborhood can be “doubled" by gluing a copy of the corner on to the original.
Doubling the domain

Doubling gives \((0, \delta) \times \mathbb{R}/2\alpha\) equipped with the metric 
\[dr^2 + r^2d\theta^2,\]
a subset of the Euclidean cone

\[C(S^1_\rho) = \mathbb{R}_+ \times \mathbb{R}/2\pi\rho,\] the Euclidean cone of radius \(\rho\) 
\((\rho = \alpha/\pi)\). It has the flat metric 
\[g = dr^2 + r^2d\theta^2.\]
Doubling the domain

- Dirichlet solutions can be extended by writing
  \[ u(t, r, \theta) = \frac{1}{\sqrt{\alpha}} \sum_{j=1}^{\infty} u_j(t, r) \sin \left( \frac{j \pi \theta}{\alpha} \right) \]

- Neumann solutions can be extended by writing
  \[ u(t, r, \theta) = u_0(t, r) + \frac{1}{\sqrt{\alpha}} \sum_{j=1}^{\infty} u_j(t, r) \cos \left( \frac{j \pi \theta}{\alpha} \right) \]
Main theorem for cones

Theorem (MDB, Ford, Marzuola)

Let $C(S^1_\rho)$ be the Euclidean cone of radius $\rho > 0$. Then for any admissible triple $(p, q, \gamma)$

$$
\|u\|_{L^p(R;L^q(C(S^1_\rho)))} \lesssim \|f\|_{\dot{H}^\gamma(C(S^1_\rho))} + \|g\|_{\dot{H}^\gamma-1(C(S^1_\rho))}
$$

- On $C(S^1_\rho)$, wave equation involves the Laplace-Beltrami operator

$$
-\Delta g = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}
$$
The Spectral Theorem

Spectral Theorem

There exists a measure space \((Y, \mu)\) and a unitary map \(W : L^2(Y, \mu) \rightarrow L^2(C(S^1_\rho))\) and a measurable function \(a(y)\) on \(Y\) such that

\[
W^{-1} \Delta g W h(y) = a(y) h(y), \quad \text{whenever } W h \in \text{Dom}(\Delta g).
\]

Furthermore, functions \(f(\Delta g)\) can be defined by

\[
W^{-1} f(\Delta g) W g(y) = f(a(y)) g(y)
\]

Can take a Littlewood-Paley decomp. w.r.t. the spectrum of \(\Delta_g\)

\[
l = \sum_{k=-\infty}^{\infty} \beta_k(\sqrt{\Delta_g})
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Functional calculus on cones

- Begin with separated solutions to the Helmholtz eqn
  \((\Delta g - \lambda^2)g(r)\varphi_\nu(\theta) = 0\), with \(-\varphi''_\nu(\theta) = \nu^2 \varphi_\nu(\theta)\)
- \(g(r)\) must satisfy the Bessel-type equation

\[
L_\nu g = -g''(r) - \frac{1}{r} g'(r) + \frac{\nu^2}{r^2} g(r) = \lambda^2 g(r) \Rightarrow g(r) = c_\nu(\lambda r)
\]

- Taking \(c_\nu(\lambda r) = J_\nu(\lambda r)\), define the Hankel transform

\[
H_\nu(g)(\lambda) = \int_0^{\infty} g(r) J_\nu(\lambda r) r \, dr
\]

- \(H_\nu\) defines a unitary map \(H_\nu : L^2(\mathbb{R}_+, r \, dr) \to L^2(\mathbb{R}_+, \lambda \, d\lambda)\)
  and \(H_\nu \circ H_\nu = I, H_\nu(L_\nu g)(\lambda) = \lambda^2 H_\nu(g)(\lambda)\)
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  and \(H_{\nu} \circ H_{\nu} = I, H_{\nu}(L_{\nu}g)(\lambda) = \lambda^2 H_{\nu}(g)(\lambda)\)
Use this to create a spectral representation of $\Delta_g$, Schwartz kernel of $f(\Delta_g)$ will have the form

$$K_f(r_1, \theta_1; r_2, \theta_2) = \sum_{\nu} \tilde{K}_f(r_1, r_2, \nu) \varphi_\nu(\theta_1) \overline{\varphi_\nu(\theta_2)}$$

where $\nu$ indexes an O.N. basis of eigenfunctions and

$$\tilde{K}_f(r_1, r_2, \nu) = \int_{0}^{\infty} f(\lambda^2) J_\nu(\lambda r_1) J_\nu(\lambda r_2) \lambda \, d\lambda$$

Use this to understand kernels of $e^{-it\sqrt{\Delta_g}}$
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Use this to understand kernels of $e^{-it\sqrt{\Delta_g}}$. 
Lipschitz-Hankel integral

\[(Q_{\nu-\frac{1}{2}} = \text{Legendre function of the 2nd kind, order } \nu - \frac{1}{2})\]

\[
\int_0^\infty e^{-it\lambda} J_\nu(\lambda r_1) J_\nu(\lambda r_2) d\lambda = \frac{1}{\pi (r_1 r_2)^{-\frac{1}{2}}} Q_{\nu-\frac{1}{2}} \left( \frac{r_1^2 + r_2^2 - t^2}{2r_1 r_2} \right)
\]

Now sum to obtain formulae for \(\sin(t \sqrt{\Delta g})/\sqrt{\Delta g}\), \(\cos(t \sqrt{\Delta g})\)

\[
K_f(r_1, \theta_1; r_2, \theta_2) = \frac{1}{2\pi\rho} \sum_{j=\infty}^{\infty} \tilde{K}_f \left( r_1, r_2, \frac{|j|}{\rho} \right) \exp \left( \frac{ij(\theta_1 - \theta_2)}{\rho} \right)
\]
Cheeger-Taylor: formulas for the kernel of $\frac{\sin(t \sqrt{\Delta_g})}{\sqrt{\Delta_g}}$ on cones

MDB-Ford-Marzuola: formulas for $\cos(t \sqrt{\Delta_g})$ when $\rho < 1$

Kernels above take the form

$$K_{geom}(r_1, \theta_1; r_2, \theta_2) + K_{diff}(r_1, \theta_1; r_2, \theta_2)$$

"$K_{geom}$" consists of terms arising from a formal application of the method of images

"$K_{diff}$" arises from diffraction by the cone tip
Formulae for the solution operators

- Cheeger-Taylor: formulas for the kernel of \( \sin(t \sqrt{\Delta_g})/\sqrt{\Delta_g} \) on cones
- MDB-Ford-Marzuola: formulas for \( \cos(t \sqrt{\Delta_g}) \) when \( \rho < 1 \)
- Kernels above take the form

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- "\( K_{\text{geom}} \)" consists of terms arising from a formal application of the method of images
- "\( K_{\text{diff}} \)" arises from diffraction by the cone tip
Good news and bad news

- Littlewood-Paley works as before and the wave equation is invariant under dilations
- Problem: Have good formulae for e.g. \( \sin( t \sqrt{\Delta_g} ) / \sqrt{\Delta_g} \), but not \( \sin( t \sqrt{\Delta_g} ) \beta_0( \sqrt{\Delta_g} ) \)
- Very difficult to obtain oscillatory integrals analogous to those on \( \mathbb{R}^2 \)
- Take a new perspective on the problem in \( \mathbb{R}^2 \) and regularize the kernel of \( \sin( t \sqrt{\Delta} ) / \sqrt{\Delta} \),

\[
K(t, x, y) = \pi^{-1} \left( t^2 - |x - y|^2 \right)^{-\frac{1}{2}}
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\]
The averaging approach on $\mathbb{R}^2$

- Treat Littlewood Paley operator as a regularizing operator

\[
\int K(t, x, y)\beta_0(\sqrt{\Delta})g_0(y) \, dy = \int \left(\beta_0(\sqrt{\Delta}y)K(t, x, y)\right) g_0(y) \, dy
\]

- On $\mathbb{R}^2$, convolution kernel of $\beta_0(\sqrt{\Delta})$ is a Schwartz function, rapidly decreasing on the unit scale.

- Morally, $|\beta_0(\sqrt{\Delta}y)K(t, x, y)|$ is controlled by its average on a set of size one.

- Averages are bounded since $(t^2 - r^2)^{-\frac{1}{2}} \leq t^{-\frac{1}{2}}(t - r)^{-\frac{1}{2}}$ and the second factor is integrable in $r$. 
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The averaging approach on $\mathbb{R}^2$

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$$\int K(t, x, y) \beta_0(\sqrt{\Delta}) g_0(y) \, dy = \int \left( \beta_0(\sqrt{\Delta} y) K(t, x, y) \right) g_0(y) \, dy$$

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- Treat Littlewood Paley operator as a regularizing operator
  \[
  \int K(t, x, y) \beta_0(\sqrt{\Delta}) g_0(y) \, dy = \int \left( \beta_0(\sqrt{\Delta_y}) K(t, x, y) \right) g_0(y) \, dy
  \]

- On $\mathbb{R}^2$, convolution kernel of $\beta_0(\sqrt{\Delta})$ is a Schwartz function, rapidly decreasing on the unit scale

- Morally, $|\beta_0(\sqrt{\Delta_y}) K(t, x, y)|$ is controlled by its average on a set of size one

- Averages are bounded since $(t^2 - r^2)^{-\frac{1}{2}} \leq t^{-\frac{1}{2}} (t - r)^{-\frac{1}{2}}$ and the second factor is integrable in $r$
The averaging approach on the Euclidean cone

Heat kernel results give bounds on the kernel of $\beta_0(\sqrt{\Delta_g})$ in terms of the distance function on $C(S^1_\rho)$, yields similar control via averages.

Behavior of the geometric term is similar to the corresponding propagator on $\mathbb{R}^2 \Rightarrow$ averaging approach carries over to the cone.

We prove pointwise bounds on the diffractive term that display a similar character:

$$|K_{diff}(r_1, \theta_1; r_2, \theta_2)| \leq (t^2 - (r_1 + r_2)^2)^{-\frac{1}{2}}$$
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