Math 581
Assignment 2, due Wednesday, February 8

1. Suppose $\mathcal{X}$ and $\mathcal{Y}$ are normed vector spaces. If $T \in B(\mathcal{X}, \mathcal{Y})$, show that

$$\begin{aligned}\sup\{\|Tx\| : \|x\| = 1\} &= \sup\left\{\frac{\|Tx\|}{\|x\|} : x \neq 0\right\} \\ &= \inf\{C : \|Tx\| \leq C\|x\| \text{ for all } x\}.\end{aligned}$$

2. Let $\mathcal{X}$ be a normed vector space and $\mathcal{M}$ a proper closed subspace of $\mathcal{X}$. Define an equivalence relation $\sim$ on $\mathcal{X}$ by declaring $x \sim y$ if and only if $x - y \in \mathcal{M}$. Denote the equivalence class of any vector $x \in \mathcal{X}$ as $[x]$ and denote the set of all equivalence classes as $\mathcal{X}/\mathcal{M}$, this is called the *quotient space*. $\mathcal{X}/\mathcal{M}$ forms a vector space with addition operation $[x]+[y] = [x+y]$ and scalar multiplication $\alpha[x] = [\alpha x]$ (check this on your own).

   (a) Show that $\|[x]\| = \inf\{\|x + y\| : y \in \mathcal{M}\}$ is a norm on $\mathcal{X}/\mathcal{M}$.

   (b) Show that if $\mathcal{X}$ is complete then so is $\mathcal{X}/\mathcal{M}$. (Hint: use exercise #1 in Assignment #1.)

   (c) Prove that the projection map $\pi : \mathcal{X} \to \mathcal{X}/\mathcal{M}$ defined by $\pi(x) = [x]$ has norm 1.

   (d) Suppose $T \in B(\mathcal{X}, \mathcal{Y})$. Prove that there is a unique $S \in B(\mathcal{X}/\mathcal{N}(T), \mathcal{Y})$ such that $T = S \circ \pi$ and $\|S\| = \|T\|$.

   (e) Prove that S as a linear mapping $S : \mathcal{X}/\mathcal{N}(T) \to \mathcal{R}(T)$, S is bijective.

   Students who have had modern algebra: find your own proofs of these theorems, avoid quoting big results from that class.

3. Suppose $\mathcal{X}$ is a Banach space. Let $T \in B(\mathcal{X})\,(= B(\mathcal{X}, \mathcal{X}))$ be an invertible operator and $S \in B(\mathcal{X})$ be such that $\|S - T\| < \|T^{-1}\|^{-1}$. Show that S is invertible and write S^{-1} in terms of a power series. Use this to prove that the set of all invertible operators in $B(\mathcal{X})$ is open. (Hint: first prove that S is invertible in the special case where T is the identity operator.)

4. A theorem from linear algebra states that if $\mathcal{X}$ is a finite dimensional vector space and $T : \mathcal{X} \to \mathcal{X}$ is linear then the following 2 implications are true:

   - If T is surjective, then T is invertible,
   - If T is injective, then T is invertible.

   Show these statements are false in general for infinite dimensional vector spaces by constructing a suitable counterexample for each statement. Easy examples can be found by considering left and right "shift" maps on $\mathbb{F}^\infty$.

5. Let $\mathcal{X}$ be a normed vector space.

   (a) Prove that if $\mathcal{M}$ is a closed subspace and $x \in \mathcal{X} \setminus \mathcal{M}$, then

   $$\mathcal{M} + \mathbb{F}x := \{y + \alpha x : y \in \mathcal{M}, \alpha \in \mathbb{F}\}$$

   is also a closed subspace.

   Hint: the natural approach is to consider a convergent sequence $\{z_n\}_1^\infty$ with $z_n \in \mathcal{M}+\mathbb{F}x$. While $z_n = y_n+\alpha_n x$ for sequences $y_n \in \mathcal{M}$, $\alpha_n \in \mathbb{F}$, it's not obvious that either of these sequences should converge! Remedy this using consequences of the Hahn-Banach theorem.

   (b) Prove that every finite dimensional subspace of $\mathcal{X}$ is closed.

Problem to do on your own (i.e. work through it but do not hand it in):

- Let $\mathcal{X}$ be a complex inner product space. For any $x, y \in \mathcal{X}$ show that the *polarization identity* holds

$$\langle x, y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right).$$