STRICHARTZ ESTIMATES FOR THE SCHRÖDINGER EQUATION
MATH 565, FALL 2017

1. The cubic nonlinear Schrödinger equation

Here we will motivate a family of estimates for the inhomogeneous Schrödinger equation known as the *Strichartz estimates*. Consider the initial value problem for the cubic nonlinear Schrödinger equation

$$i\partial_t v + \Delta v = \pm |v|^2 v, \qquad v(0,x) = h(x). \tag{1.1}$$

In Assignment 7, you will show that for solutions $v(t,x)$ whose derivatives of up to order 2 tend to 0 as $|x| \to \infty$, the following quantities are conserved (i.e. they are independent of t):

$$E[v](t) = \int_{\mathbb{R}^n} \frac{1}{2}|\nabla_x v(t,x)|^2 \pm \frac{1}{4}|v(t,x)|^4 dx, \tag{1.2}$$

$$M[v](t) = \frac{1}{2}\int_{\mathbb{R}^n} |v(t,x)|^2\, dx. \tag{1.3}$$

The choice of $+$ in (1.1) is often called the "defocusing" case and the choice of $-$ is called the "focusing" case. Note that in the former case, the quantity $E[v]$ is always positive, whereas in the latter, this is not necessarily the case. This turns out to have a considerable impact on the behavior of solutions.

As with many PDE's one is often interested in the *well-posedness* of these equations, that is, the existence and uniqueness of solutions along with continuous dependence on initial data. A topic of considerable interest is to develop a well-posedness for these equations with *rough initial data*, which means considering initial data h in (1.1) which is not necessarily infinitely differentiable, rather it lies in a Sobolev space or other function space. There are a number of reasons for wanting to develop such a theory, the most immediate is that since the conserved quantities (1.2), (1.3) involve derivatives of order at most 1, then establishing such a theory will allow for an efficient use of these quantities, which can be important for determining if "lifespan" of solutions, i.e. does $v(t,\cdot)$ exist for all t or just in some locally for t in a finite interval containing 0? Moreover, there are related questions such as *persistence of regularity*: if $h \in H^s(\mathbb{R}^n)$, then is $v(t,\cdot) \in H^s(\mathbb{R}^n)$ for all t in the lifespan? Finally, *scattering* phenomena is often of interest: if the lifespan of v is all of $\mathbb{R}$, is it possible to say that $v(t,\cdot)$ in some sense behaves like a solution to the free equation $i\partial_t w + \Delta w = 0$ as $t \to \pm\infty$?

This of course raises the question of what a solution to (1.1) is when $h \notin C^2$. In this case, we consider v to be a solution to the integral equation

$$v(t,x) = (e^{it\Delta}h)(x) \mp i\int_0^t \big(e^{i(t-s)\Delta}|v(s,\cdot)|^2 v(s,\cdot)\big)(x)\, ds. \tag{1.4}$$

In the past, we typically have only applied the operator $e^{it\Delta}$ to functions in $\mathcal{S}(\mathbb{R}^n)$, but it is a well defined operator on any Sobolev space, and it is not hard to see that $\|e^{it\Delta}h\|_{H^s} = \|h\|_{H^s}$ for any s and the same holds for the homogeneous counterparts. Moreover, since $\mathcal{S}(\mathbb{R}^n)$ is dense in H^s, and dense in $\dot{H}^s$ for $s > -\frac{n}{2}$, then very often we it suffices to consider $h \in \mathcal{S}(\mathbb{R}^n)$, then only reference the H^s norms of h and of the solution.

One effective strategy to this end is to treat the right hand side $\pm|v|^2 v$ in (1.1) as introducing a small perturbation of the free equation $i\partial_t w + \Delta w = 0$. At first glance this may seem surprising, but if the solution at some time satisfies $|v(t,x)| \ll 1$, then $|v(t,x)|^3 \ll |v(t,x)|$ and so the contribution

of the nonlinearity is in some sense smaller. But if this strategy in Sobolev spaces is to be effective then we need to limit the strength of the nonlinearity. As a concrete example of this, suppose we want to consider (1.1) when $n = 2$, so that $(t, x) \in \mathbb{R} \times \mathbb{R}^2$ and that we want to assume $h \in L^2(\mathbb{R}^2)$. In this case, the L^2 bounds for the Schrödinger equation (the effect of applying the L^2 bounds for $e^{it\Delta}$ to (1.4)) yield

$$\|v(t,\cdot)\|_2 \le \|h\|_2 + \int_0^t \left\| |v(s,\cdot)|^2 v(s,\cdot) \right\|_2 ds = \|h\|_2 + \int_0^t \|v(s,\cdot)\|_6^3 \, ds.$$

This means considering v in the mixed-norm space $L^3([0,t]; L^6(\mathbb{R}^2))$. More generally for exponents p, q and an interval $(-T, T) \subset \mathbb{R}$, we define

$$\|v\|_{L^p_T L^q} := \left(\int_{-T}^T \|v(t,\cdot)\|_q^p dt \right)^{1/p} = \left(\int_{-T}^T \left(\int_{\mathbb{R}^n} |v(t,x)|^q \, dx \right)^{p/q} dt \right)^{1/p}, \tag{1.5}$$

whenever the dimension n is understood. This is the natural norm on the vector-valued L^p space $L^p((-T,T); L^q(\mathbb{R}^n))$. Consequently, the norms we encounter involve space-time integrals, in contrast to those where t is fixed. When $T = \infty$, we denote this more simply as $L^p L^q$, $\|v\|_{L^p L^q} := \left(\int_{-\infty}^{\infty} \|v(t,\cdot)\|_q^p dt \right)^{1/p}$.

One thing we have been building up to is to prove the following inequality for solutions to the inhomogeneous Schrödinger equation $i\partial_t u + \Delta u = F$ on $\mathbb{R} \to \mathbb{R}^2$

$$\|u\|_{L^4((-T,T)\times\mathbb{R}^2)} + \sup_{t\in\mathbb{R}} \|u(t,\cdot)\|_{L^2(\mathbb{R}^2)} \le C_0 \left(\|u(0,\cdot)\|_{L^2(\mathbb{R}^2)} + \|F\|_{L^{4/3}((-T,T)\times\mathbb{R}^2)} \right) \tag{1.6}$$

where C_0 is independent of T. This does not involve the mixed-norms in (1.5), but does offer an elegant way to obtain local well-posedness of (1.1) when $n = 2$. To see this, we will apply the principle in Exercise #4 in Assignment 7, taking $\mathcal{X} = L^4((-T,T)\times\mathbb{R}^2)$, $\mathcal{Y} = L^{4/3}((-T,T)\times\mathbb{R}^2)$ for some $T > 0$, and $\mathbb{Z} = L^2(\mathbb{R}^2)$. Considering solutions to the inhomogeneous equation with vanishing initial data, (1.6) shows that

$$\left\| \int_0^t \left(e^{i(t-s)\Delta} F(s,\cdot) \right) ds \right\|_{L^4((-T,T)\times\mathbb{R}^2)} \le C_0 \|F\|_{L^{4/3}((-T,T)\times\mathbb{R}^2)}.$$

In other words, $\|\mathcal{D}F\|_{L^4((-T,T)\times\mathbb{R}^2)} \le C_0 \|F\|_{L^{4/3}((-T,T)\times\mathbb{R}^2)}$. Moreover, since the constant C_0 in (1.6) can be taken independently of T,

$$\|e^{it\Delta} h\|_{L^4(\mathbb{R}\times\mathbb{R}^2)} = \sup_{T>0} \|e^{it\Delta} h\|_{L^4((-T,T)\times\mathbb{R}^2)} \le C_0 \|h\|_{L^2(\mathbb{R}^2)}.$$

Hence by the monotone convergence theorem

$$\lim_{T\to 0+} \|e^{it\Delta} h\|_{L^4((-T,T)\times\mathbb{R}^2)} = 0.$$

Thus given $r > 0$, we can take T sufficiently small so that $\|e^{it\Delta} h\|_{L^4((-T,T)\times\mathbb{R}^2)} < r/2$. Finally, observe that

$$|v_1|^2 v_1 - |v_2|^2 v_2 = |v_1|^2 (v_1 - v_2) + v_2 \overline{v_1} (v_1 - v_2) + v_2^2 (\overline{v_1} - \overline{v_2})$$

hence by the generalized Hölder inequality, using that $\frac{3}{4} = \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$ (Assignment 2, Exercise 5)

$$\begin{aligned} &\left\| |v_1|^2 v_1 - |v_2|^2 v_2 \right\|_{L^{4/3}((-T,T)\times\mathbb{R}^2)} \\ &\qquad \le 3 \max\left(\|v_1\|^2_{L^4((-T,T)\times\mathbb{R}^2)}, \|v_2\|^2_{L^4((-T,T)\times\mathbb{R}^2)} \right) \|v_1 - v_2\|_{L^4((-T,T)\times\mathbb{R}^2)}. \end{aligned}$$

Therefore if $\|v_j\|_{L^4((-T,T)\times\mathbb{R}^2)} < r$ for $j = 1, 2$ we have

$$\left\| |v_1|^2 v_1 - |v_2|^2 v_2 \right\|_{L^{4/3}((-T,T)\times\mathbb{R}^2)} < 3r^2 \|v_1 - v_2\|_{L^4((-T,T)\times\mathbb{R}^2)}.$$

We now see that if r is chosen so that $3r^2 < (2C_0)^{-1}$, the contraction principle shows that we have a solution v to the integral equation (1.4). We did not make use of the quantity $\sup_{t\in(-T,T)} \|u(t,\cdot)\|_{L^2(\mathbb{R}^2)}$ on the left hand side of (1.6), but this turns out to have other uses.

2. Strichartz estimates

Theorem 2.1. *Suppose u is a solution to the inhomogeneous Schrödinger equation $i\partial_t u + \Delta u = F$ on $\mathbb{R}\times\mathbb{R}^n$ with initial data $u(0,\cdot) = h \in L^2(\mathbb{R}^n)$. Suppose further that*

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2}, \qquad 2 < p \le \infty,\ 2 \le q \le \infty, \tag{2.1}$$

$$\frac{2}{\tilde p} + \frac{n}{\tilde q} = \frac{n}{2}, \qquad 2 < \tilde p \le \infty,\ 2 \le \tilde q \le \infty, \tag{2.2}$$

$$2 \le \tilde q \le q. \tag{2.3}$$

Then there exists a constant C depending only on $p, q, \tilde p, \tilde q$ such that

$$\|u\|_{L^p_T L^q} \le C\left(\|h\|_2 + \|F\|_{L^{\tilde p'}_T L^{\tilde q'}}\right), \tag{2.4}$$

provided of course the right hand side is finite. In particular, since C is independent of T, we have

$$\|u\|_{L^p L^q} \le C\left(\|h\|_2 + \|F\|_{L^{\tilde p'} L^{\tilde q'}}\right), \tag{2.5}$$

Simple scaling considerations show that in general, one cannot relax the conditions on the exponents in (2.1), (2.2). To see this, consider a homogeneous solution $(i\partial_t + \Delta)u(t,x) = 0$ on $\mathbb{R}\times\mathbb{R}^n$ and correspondingly define

$$u_\lambda(t,x) = u(\lambda^2 t, \lambda x), \quad h_\lambda(x) = h(\lambda x),$$

so that $u_\lambda(0,x) = h(x)$ and the chain rule yields $(i\partial_t + \Delta)u_\lambda(t,x) = 0$. Now compute $\|u_\lambda\|_{L^pL^q}$:

$$\|u_\lambda\|_{L^pL^q} = \left(\int_{-\infty}^{\infty}\left(\int_{\mathbb{R}^n} |u_\lambda(t,x)|^q\,dx\right)^{\frac{p}{q}} dt\right)^{\frac{1}{p}} = \lambda^{-\frac{2}{p}-\frac{n}{q}}\left(\int_{-\infty}^{\infty}\left(\int_{\mathbb{R}^n} |u(s,y)|^q\,dy\right)^{\frac{p}{q}} ds\right)^{\frac{1}{p}}$$

via the substitution $s = \lambda^2 t$, $y = \lambda x$. We thus have $\|u_\lambda\|_{L^pL^q} = \lambda^{-\frac{2}{p}-\frac{n}{q}}\|u\|_{L^pL^q}$. A similar computation shows that $\|h_\lambda\|_{L^2(\mathbb{R}^n)} = \lambda^{-\frac{n}{2}}\|h\|_{L^2(\mathbb{R}^n)}$. Thus if a bound of the form (2.5) holds for all solutions to the homogeneous Schrodinger equation we have that $\|u_\lambda\|_{L^pL^q} \le C\|h_\lambda\|_{L^2(\mathbb{R}^n)}$

$$\|u\|_{L^pL^q} \le C\lambda^{\frac{2}{p}+\frac{n}{q}-\frac{n}{2}}\|h\|_{L^2(\mathbb{R}^n)}.$$

Thus if we had $\frac{2}{p} + \frac{n}{q} - \frac{n}{2} > 0$, we could take the limit as $\lambda \to 0$ to see that $u \equiv 0$ even though $h \ne 0$, a contradiction. The same conclusion holds if $\frac{2}{p} + \frac{n}{q} - \frac{n}{2} < 0$, but this time taking the limit $\lambda \to \infty$. A very similar computation shows that (2.2) must hold for the inhomogeneous equation when F is nontrivial.

Lemma 2.2. *For each $t \in \mathbb{R}$, the operator $e^{it\Delta}$ maps $L^{q'}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ with*

$$\|e^{it\Delta}h\|_{L^q(\mathbb{R}^n)} \le (4\pi|t|)^{\frac{n}{q}-\frac{n}{2}}\|h\|_{L^{q'}(\mathbb{R}^n)}. \tag{2.6}$$

Proof. The lemma is very similar to the proof of the Hausdorff-Young inequality from the notes on interpolation. Recall that

$$\|e^{it\Delta}h\|_{L^\infty(\mathbb{R}^n)} \le (4\pi|t|)^{-\frac{n}{2}}\|h\|_{L^1(\mathbb{R}^n)},$$
$$\|e^{it\Delta}h\|_{L^2(\mathbb{R}^n)} = \|h\|_{L^2(\mathbb{R}^n)}.$$

We now write

$$\frac{1}{q} = \frac{1-t}{2} + \frac{t}{\infty} \quad \text{and} \quad \frac{1}{p} = \frac{1-t}{2} + \frac{t}{1}, \tag{2.7}$$

so that (p^{-1}, q^{-1}) lies on the line segment joining the points $(0,1)$ and $(2,2)$ in $\mathbb{R}^2$. As in the proof of Hausdorff-Young, these two points also lie on the line $\frac{1}{\tilde{q}} = 1 - \frac{1}{\tilde{p}}$ in the $(\tilde{p}^{-1}, \tilde{q}^{-1})$ plane, in other words, they lie on the line which determines Hölder conjugates. Consequently, p, q as defined in (2.7) are Hölder conjugates. The inequality (2.6) then follows from the Riesz-Thorin interpolation theorem with $M_0 = 1$, $M_1 = (4\pi|t|)^{-\frac{n}{2}}$ and solving for t in (2.7) yields $t = 1 - \frac{2}{q}$ so that $M_0^{1-t} M_1^t = (4\pi|t|)^{-\frac{n}{2}(1-\frac{2}{q})}$. □

Proof of Theorem 2.1. Initially, we concern ourselves with the bounds (2.4) in the homogeneous case, i.e. the $F \equiv 0$ case. Let $\mathcal{T}$ be the operator which acts on function $h \in L^2(\mathbb{R}^n)$ via

$$(\mathcal{T}h)(t,x) = (e^{it\Delta}h)(x) = \mathcal{F}^{-1}(e^{-it|\xi|^2}\widehat{h}(\xi)) = \int_{\mathbb{R}^n} K_t(x-y)h(y)\,dy, \qquad K_t(y) := (4\pi it)^{-\frac{n}{2}}\, e^{\frac{i}{4t}|y|^2},$$

where we have used the result in Exercise 4a of Assignment 4. Hence $\mathcal{T}$ maps a function of n variables to one of $n+1$ variables. When $F \equiv 0$, (2.4) reads as

$$\|\mathcal{T}h\|_{L^p_T L^q} \le C\|h\|_2, \qquad \frac{2}{p} + \frac{n}{q} = \frac{n}{2}. \tag{2.8}$$

We begin by computing its adjoint $\mathcal{T}^*$, in other words the function $\langle \mathcal{T}h, G\rangle = \langle h, \mathcal{T}^*G\rangle$ for functions $G(t,x) \in L^2((-T,T)\times\mathbb{R}^n)$ with respect to the L^2 inner products on $(-T,T)\times\mathbb{R}^n$ and $\mathbb{R}^n$ respectively:

$$\begin{aligned}
\langle \mathcal{T}h, G\rangle &= \int_{-T}^{T}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} K_t(x-y)h(y)\,dy\right)\overline{G(t,x)}\,dxdt \\
&= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\int_{-T}^{T} K_t(x-y)\overline{G(t,x)}\,dtdx\right)h(y)\,dy \\
&= \int_{\mathbb{R}^n}\overline{\left(\int_{\mathbb{R}^n}\int_{-T}^{T} K_{-t}(y-x)G(t,x)\,dtdx\right)}h(y)\,dy
\end{aligned}$$

where we have used that $\overline{K_t(y)} := \overline{(4\pi it)^{-\frac{n}{2}}\, e^{\frac{i}{4t}|y|^2}} = (-4\pi it)^{-\frac{n}{2}}\, e^{-\frac{i}{4t}|y|^2} = K_{-t}(y)$ and that the function is radial in y. This computation shows that the adjoint must satisfy

$$(\mathcal{T}^*G)(y) = \int_{-T}^{T}\int_{\mathbb{R}^n} K_{-t}(y-x)G(t,x)\,dxdt = \int_{-T}^{T}\left(e^{-it\Delta}G(t,\cdot)\right)(y)dt,$$

and hence $\mathcal{T}^*$ caries functions of $n+1$ variables to those in n variables.

The bound (2.8) follows from the following inequality for $\mathcal{T}^*$

$$\|\mathcal{T}^*G\|_{L^2(\mathbb{R}^n)} \le C\|G\|_{L^{p'}L^{q'}}. \tag{2.9}$$

Indeed, since

$$\|\mathcal{T}h\|_{L^p_T L^q} = \sup\left\{\left|\int_{-T}^{T}\int_{\mathbb{R}^n}\mathcal{T}h(t,x)\overline{G(t,x)}\,dxdt\right| : \|G\|_{L^{p'}_T L^{q'}} = 1\right\}$$

then if (2.9) holds, we will have for $\|G\|_{L^{p'}_T L^{q'}} = 1$

$$\left|\int_{-T}^{T}\int_{\mathbb{R}^n}\mathcal{T}h(t,x)\overline{G(t,x)}\,dxdt\right| = |\langle \mathcal{T}h, G\rangle| = |\langle h, \mathcal{T}^*G\rangle| \le \|h\|_{L^2(\mathbb{R}^n)}\|\mathcal{T}^*G\|_{L^2(\mathbb{R}^n)} \le C\|h\|_{L^2(\mathbb{R}^n)}$$

The next observation is that

$$\|\mathcal{T}^*G\|^2_{L^2(\mathbb{R}^n)} = \langle \mathcal{T}^*G, \mathcal{T}^*G\rangle = \langle G, \mathcal{T}\mathcal{T}^*G\rangle \le \|G\|_{L^{p'}_T L^{q'}}\|\mathcal{T}\mathcal{T}^*G\|_{L^p_T L^q}$$

where the last inequality is a consequence of Hölder's inequality. Consequently if we can show that there exists a constant C such that

$$\|\mathcal{T}\mathcal{T}^* G\|_{L^p_T L^q} \leq C\|G\|_{L^{p'}_T L^{q'}}, \tag{2.10}$$

then (2.9) will follow. Moving forward, we will use that

$$\mathcal{T}\mathcal{T}^* G(t,\cdot) = e^{it\Delta}\mathcal{T}^* G = e^{it\Delta}\left(\int_{-T}^{T}\left(e^{-is\Delta}G(s,\cdot)\right)ds\right) = \int_{-T}^{T}\left(e^{i(t-s)\Delta}G(s,\cdot)\right)ds. \tag{2.11}$$

We are reduced to showing (2.10), and it suffices to consider $G(t,x)$ satisfying $\|G\|_{L^{p'}_T L^{q'}} = 1$. By Minkowski's inequality for integrals

$$\|\mathcal{T}\mathcal{T}^* G(t,\cdot)\|_{L^q} \leq \int_{-T}^{T}\left\|\left(e^{i(t-s)\Delta}G(s,\cdot)\right)\right\|_{L^q} ds \leq \int_{-\infty}^{\infty}(4\pi|t-s|)^{\frac{n}{q}-\frac{n}{2}}\|G(s,\cdot)\|_{L^{q'}}\mathbf{1}_{(-T,T)}(s)\,ds.$$

We now define α so that $\frac{n}{q}-\frac{n}{2} = \alpha - 1$, and the condition on p,q in (2.8) implies that $\alpha = 1-\frac{2}{p}$. The previous inequality implies

$$\|\mathcal{T}\mathcal{T}^* G\|_{L^p_T L^q} \leq C\left(\int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty}(4\pi|t-s|)^{\alpha-1}\|G(s,\cdot)\|_{L^{q'}}\mathbf{1}_{(-T,T)}(s)\,ds\right|^{\frac{p}{q}}dt\right)^{\frac{1}{p}},$$

but based on our definition of α we have that $\frac{1}{p} = \frac{1}{p'} - \alpha$. Hence the Hardy-Littlewood-Sobolev theorem in 1 dimension yields that the right hand side is bounded above by

$$\left(\int_{-\infty}^{\infty}\|G(s,\cdot)\|^{p'}_{L^{q'}(\mathbb{R}^n)}\mathbf{1}_{(-T,T)}(s)\,ds\right)^{\frac{1}{p'}} = \left(\int_{-T}^{T}\|G(s,\cdot)\|^{p'}_{L^{q'}(\mathbb{R}^n)}ds\right)^{\frac{1}{p'}} = \|G\|_{L^{p'}_T L^{q'}} = 1,$$

which concludes the proof of (2.10) and hence (2.4) when $F \equiv 0$. Note that the constant is indeed independent of T since it only depends on the constant in (2.6) and the one appearing in the Hardy-Littlewood-Sobolev theorem.

At this point, the case where F is nontrivial "should" follow since the structure of $\mathcal{T}\mathcal{T}^*$ in (2.11) is similar to the structure of the Duhamel term

$$\mathcal{D}F(t,\cdot) = \int_0^t\left(e^{i(t-s)\Delta}F(s,\cdot)\right)ds, \tag{2.12}$$

and we now know that $\mathcal{T} : L^2(\mathbb{R}^n) \to L^p_T L^q$ and $\mathcal{T}^* : L^{\tilde{p}'}_T L^{\tilde{q}'} \to L^2(\mathbb{R}^n)$ for any choice of p,q and $\tilde{p},\tilde{q}$ in (2.1), (2.2), and hence with constants changing in each inequality

$$\left\|\int_{-T}^{T}\left(e^{i(t-s)\Delta}G(s,\cdot)\right)ds\right\|_{L^p_T L^q} = \|\mathcal{T}\mathcal{T}^* G\|_{L^p_T L^q} \leq C\|\mathcal{T}^* G\|_{L^2(\mathbb{R}^n)} \leq C\|G\|_{L^{\tilde{p}'}_T L^{\tilde{q}'}} \tag{2.13}$$

However, the expressions for $\mathcal{D}$ and $\mathcal{T}\mathcal{T}^*$ are not identical since the upper limit of integration depends on t here and the lower limit is 0 instead of $-T$. The latter isn't really much of a problem since we can always replace $F(s,\cdot)$ by $F(s,\cdot)\mathbf{1}_{(0,t)\cup(t,0)}(s)$ (treating e.g. $(0,t) = \emptyset$ when $t \leq 0$):

$$\int_0^t\left(e^{i(t-s)\Delta}F(s,\cdot)\right)ds = \int_{-T}^{T}\left(e^{i(t-s)\Delta}F(s,\cdot)\right)\mathbf{1}_{(0,t)\cup(t,0)}(s)\mathrm{sgn}(s)\,ds$$

However, the time dependence in the upper limit of integration does indeed effect the analysis in a subtle way.

Nonetheless we can conclude (2.4) for exponents $2 \leq \tilde{q} \leq q$ by interpolation. To see this first observe that

$$\begin{aligned}\left\| \int_{-T}^{T} \left(e^{i(t-s)\Delta} F(s,\cdot)\right) \mathbf{1}_{(0,t)\cup(t,0)}(s) \mathrm{sgn}(s)\, ds \right\|_{L^q} &\leq \int_{-T}^{T} \left\| \left(e^{i(t-s)\Delta} F(s,\cdot)\right) \right\|_{L^q} \mathbf{1}_{(0,t)\cup(t,0)}(s)\, ds \\ &\leq \int_{-T}^{T} (4\pi|t-s|)^{\frac{n}{q}-\frac{n}{2}} \left\| F(s,\cdot) \right\|_{L^{q'}}\, ds.\end{aligned}$$

Integrating both sides of this inequality over $L^p(-T,T)$ and the previous application of Hardy-Littlewood-Sobolev then yields for some constant C independent of T

$$\left\| \int_0^t \left(e^{i(t-s)\Delta} F(s,\cdot)\right) ds \right\|_{L^p_T L^q} \leq C \|F\|_{L^{p'}_T L^{q'}}. \tag{2.14}$$

At the same time since $|\mathbf{1}_{(0,t)\cup(t,0)}(s)\mathrm{sgn}(s)| \leq 1$,

$$\begin{aligned}\left\| \int_0^t \left(e^{i(t-s)\Delta} F(s,\cdot)\right) ds \right\|_{L^p_T L^q} &\leq \int_{-T}^{T} \left\| \mathcal{T}\left(e^{-is\Delta} F(s,\cdot)\right) \right\|_{L^p_T L^q}\, ds \\ &\leq \int_{-T}^{T} \left\| e^{-is\Delta} F(s,\cdot) \right\|_2\, ds = \int_{-T}^{T} \left\| F(s,\cdot) \right\|_2\, ds\end{aligned}$$

In summary,

$$\left\| \int_0^t \left(e^{i(t-s)\Delta} F(s,\cdot)\right) ds \right\|_{L^p_T L^q} \leq C \|F\|_{L^1_T L^2}. \tag{2.15}$$

If $\frac{2}{\tilde{p}} + \frac{n}{\tilde{q}} = \frac{n}{2}$ and $2 < \tilde{q} < q$, we can then apply Riesz-Thorin interpolation to (2.14) and (2.15) to obtain the remaining bounds (2.4). □

Remark 2.3. *The restriction* (2.3) *can actually be removed by using what is now known as the Christ-Kiselev Lemma. In a nutshell, this allows one to see that the bound on the Duhamel term in* (2.12)

$$\left\| \int_0^t \left(e^{i(t-s)\Delta} F(s,\cdot)\right) ds \right\|_{L^p_T L^q} \leq C \|F\|_{L^{\tilde{p}'}_T L^{\tilde{q}'}}$$

does indeed follow from the one in (2.13)*:*

$$\left\| \int_{-T}^{T} \left(e^{i(t-s)\Delta} G(s,\cdot)\right) ds \right\|_{L^p_T L^q} \leq C \|G\|_{L^{\tilde{p}'}_T L^{\tilde{q}'}}.$$

Another achievement in the theory of Strichartz estimates is the endpoint estimate of Keel and Tao, which allows one to take $p = 2$ or $\tilde{p} = 2$ above. Indeed, the arguments above using the Hardy-Littlewood-Sobolev theorem break down when $p = 2$ since this means that $\alpha = 0$, at which point the fractional integration bound fails to hold. So other methods are needed to see that this holds.