Math 565, Fall 2017
Assignment 2, due Wednesday, September 20

Hand in solutions to the following problems:

1. Use convolutions to prove the "C^∞ Urysohn Lemma": If $K \subset \mathbb{R}^n$ is compact and U is an open set containing K, then there exists $\psi \in C_c^\infty(\mathbb{R}^n)$ such that $0 \le \psi(x) \le 1$ for every $x \in \mathbb{R}^n$, $\psi(x) = 1$ for every $x \in K$, and $\operatorname{supp}(\psi) \subset U$. You may need to use the standard 510 result that $d(U^C, K) > 0$.

2. Young's inequality for products via differential calculus: Prove that for any $a, b > 0$ and Hölder conjugates $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$,

$$ab \le \frac{a^p}{p} + \frac{b^q}{q},$$

   by fixing b and proving the existence of a global maximum for $f(a) = ab - \frac{a^p}{p}$ on $(0, \infty)$. Conclude that equality is satisfied if and only if $a^p = b^q$.

3. Prove that if $0 < p < q < r \le \infty$, then $L^q(\mathbb{R}^n) \subset L^p(\mathbb{R}^n) + L^r(\mathbb{R}^n)$; that is, each $f \in L^q(\mathbb{R}^n)$ can be written as $f = g + h$, the sum of a function in $g \in L^p(\mathbb{R}^n)$ and a function in $h \in L^r(\mathbb{R}^n)$.

4. Fix an arbitrary p_0, p_1 with $0 < p_0 < p_1 < \infty$. Find examples of continuous functions f on $(0, \infty)$, such that $f \in L^p(0, \infty)$ if and only if $p_0 < p < p_1$. Hint: consider piecewise defined functions involving x^{-a}.

5. Prove the following generalization of Hölder's inequality: Suppose that

$$\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_k} = \frac{1}{r}$$

   with $1 \le p_j \le \infty$ for $j = 1, \ldots, k$ and $1 \le r \le \infty$. If $f_j \in L^{p_j}(\mathbb{R}^n)$ for $j = 1, \ldots, k$, then $\Pi_{j=1}^k f_j \in L^r(\mathbb{R}^n)$ and

$$\|\Pi_{j=1}^k f_j\|_r \le \Pi_{j=1}^k \|f_j\|_{p_j}$$

   Hint: using induction, reduce to the case where $k = 2$. Then derive this as a consequence of the usual Hölder inequality.

6. (a) Suppose $f \in L^p(\mathbb{R}^n)$ for some $1 < p < \infty$. Prove that with $\frac{1}{p} + \frac{1}{q} = 1$,

$$\|f\|_p = \max\left\{ \left| \int_{\mathbb{R}^n} f(x) g(x)\, dx \right| : \|g\|_q = 1 \right\}.$$

   Hint: consider g of the form $g(x) = C\,\overline{\operatorname{sgn}(g(x))}|g(x)|^{p-1}$ for some constant C.

   (b) Prove Minkowski's inequality for integrals: Suppose $f : \mathbb{R}^{l+m} = \mathbb{R}^l \times \mathbb{R}^m \to \mathbb{C}$ is a measurable function. Then

$$\left(\int_{\mathbb{R}^l} \left| \int_{\mathbb{R}^m} f(x,y)\, dy \right|^p dx \right)^{\frac{1}{p}} \le \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^l} |f(x,y)|^p\, dx \right)^{\frac{1}{p}} dy,$$

   provided the right hand side is finite. Hint: first reduce to the case where f is nonnegative and measurable. Then use part (a) and the Tonelli theorem on iterated integrals.

Note on #6: Strictly speaking there are issues in deducing the measurability of the slice functions $f(x,\cdot):\mathbb{R}^m\to\mathbb{C}$ for a fixed $x\in\mathbb{R}^l$ and $f(\cdot,y):\mathbb{R}^l\to\mathbb{C}$ for a fixed $y\in\mathbb{R}^m$. This is typically discussed in detail in measure theory courses. You do not need to address such matters here. In short, this typically does not present any difficulty since the measurability of the slice functions hold up to a set of measure zero.

On your own:

1. Use the result in #4 from Assignment #1 to show that if $a>0$ and $f(x)=\exp(-a|x|^2)$ for $x\in\mathbb{R}^n$ then $\widehat{f}(\xi)=\left(\frac{\pi}{a}\right)^{n/2}\exp(-|\xi|^2/4a)$. (It is not hard to see that the n-dimensional case easily reduces to that in 1 dimension.)

   It is interesting to note that there is another proof of this when $n=1$, considering the ordinary differential equation satisfied by $\widehat{f}$ and integrating factors.

2. Revisit problem #4 and find examples of functions f such that $f\in L^p(0,\infty)$ if and only if

   (a) $p_0\le p\le p_1$

   (b) $p=p_0$

   Hint: This time consider piecewise defined functions involving $x^{-a}|\log x|^{-b}$.

3. Suppose $0<p<1$. Prove that for any $a,b>0$, $(a+b)^p<a^p+b^p$. Consider disjoint sets $E,F\subset\mathbb{R}^n$ of positive measure (and their respective indicator functions) to conclude that the triangle inequality

$$\left(\int_{\mathbb{R}^n}|f(x)+g(x)|^p dx\right)^{\frac{1}{p}}\le\left(\int_{\mathbb{R}^n}|f(x)|^p dx\right)^{\frac{1}{p}}+\left(\int_{\mathbb{R}^n}|g(x)|^p dx\right)^{\frac{1}{p}}$$

   is generally false when $0<p<1$.

   Hint: $f(t)=pt^{p-1}-p(t+a)^{p-1}$ is a positive, improperly integrable function on $[0,b]$.