FOURIER TRANSFORMS OF SURFACE MEASURE ON THE SPHERE
MATH 565, FALL 2017

1. Fourier transform of surface measure on the sphere

Recall that the distribution u on $\mathbb{R}^n$ defined as

$$\langle u, \psi \rangle = \int_{\mathbb{S}^{n-1}} \psi(x)\, d\sigma(x)$$

is compactly supported and hence defines a tempered distribution on $\mathbb{R}^n$. Here $d\sigma$ denotes the usual surface measure on $\mathbb{S}^{n-1}$. Indeed, it is apparent that if $\mathrm{supp}(\psi) \cap \mathbb{S}^{n-1} = \emptyset$, then $\langle u, \psi \rangle = 0$. As a consequence of our theorem on Fourier transforms of compactly supported distributions, we have that the distribution $\widehat{u}$ is given by integration against the C^∞ slowly increasing function

$$\widehat{u}(\xi) = \int_{\mathbb{S}^{n-1}} e^{-ix\cdot\xi}\, d\sigma(x). \tag{1.1}$$

Theorem 1.1. *Suppose $\widehat{u}(\xi)$ is as in (1.1). Then for $|\xi| \geq 1$,*

$$\widehat{u}(\xi) = 2(2\pi)^{\frac{n-1}{2}} |\xi|^{-\frac{n-1}{2}} \cos\left(|\xi| - \frac{\pi(n-1)}{4}\right) + R(\xi) \tag{1.2}$$

where $R(\xi) = \mathcal{O}(|\xi|^{-\frac{n+1}{2}})$ as $|\xi| \to \infty$, that is, $|R(\xi)| \leq C|\xi|^{-\frac{n+1}{2}}$ for some constant C.

Proof. We begin by observing that is $\widehat{u}(\xi)$ is radial. Indeed, given any orthogonal matrix A,

$$\widehat{u}(A\xi) = \int_{\mathbb{S}^{n-1}} e^{-ix\cdot(A\xi)}\, d\sigma(x) = \int_{\mathbb{S}^{n-1}} e^{-iA^T x\cdot\xi}\, d\sigma(x) = \int_{\mathbb{S}^{n-1}} e^{-iz\cdot\xi}\, d\sigma(z) = \widehat{u}(\xi)$$

where in the second to last identity we used that $d\sigma$ is invariant under the change of variables $z = A^T x$. Thus in what follows, we may assume that $\xi = \lambda e_n$, where $e_n = (0, \ldots, 0, 1)$ is the n-th standard basis vector. We stress that this means $|\xi| = \lambda$. We are then left to compute

$$\widehat{u}(\lambda e_n) = \int_{\mathbb{S}^{n-1}} e^{-i\lambda x_n}\, d\sigma(x). \tag{1.3}$$

We now parameterize $\mathbb{S}^{n-1}$ by letting y denote variables in $\mathbb{R}^{n-1}$ and viewing the upper and lower hemispheres as a graph $y \mapsto (y, \pm\sqrt{1-|y|^2})$ over the unit ball $B(0,1) \subset \mathbb{R}^{n-1}$. This yields

$$\widehat{u}(\lambda e_n) = \sum_{\pm} \int_{B(0,1)} e^{\mp i\lambda\sqrt{1-|y|^2}} (1-|y|^2)^{-1/2}\, dy. \tag{1.4}$$

Here we have used that

$$\nabla\left(\sqrt{1-|y|^2}\right) = \frac{-y}{\sqrt{1-|y|^2}}, \quad \text{so that } \sqrt{1 + |\nabla(\sqrt{1-|y|^2})|^2} = (1-|y|^2)^{-1/2}. \tag{1.5}$$

Hence this last expression appears in (1.4) on account of the usual formula for the surface measure element determined by a graph in $\mathbb{R}^n$. The first expression here shows that the only critical points of the oscillatory integral defining $\widehat{u}$ occur at $y = 0$ which in turn corresponds to the north and south poles. Note that moreover, (1.4) implies that

$$\widehat{u}(\lambda e_n) = 2\mathrm{Re}\left(\int_{B(0,1)} e^{i\lambda\sqrt{1-|y|^2}} \frac{1}{\sqrt{1-|y|^2}}\, dy\right),$$

showing that $\widehat{u}(\xi)$ is always real valued. Note that this is almost the real part of an "$I(\lambda)$" form that we typically like to write, the only difference being that the amplitude functions are not C^∞ here. This does not turn out to be of much significance since away from a neighborhood of the origin, we have a phase function with no critical points.

We are now led to define

$$I(\lambda) = \int e^{i\lambda\phi(y)}\psi(y)(1-|y|^2)^{-1/2}\,dy \tag{1.6}$$

where $\psi \in C_c^\infty(B(0,1/2))$ satisfies $\phi = 1$ on $B(0,1/4)$ and $\phi(y) = \sqrt{1-|y|^2}$. We then have

$$\widehat{u}(\lambda e_n) = 2\mathrm{Re}\,(I(\lambda)) + \sum_{\pm}\int_{B(0,1)} e^{\mp i\lambda\sqrt{1-|y|^2}}(1-\psi)(y)(1-|y|^2)^{-1/2}\,dy. \tag{1.7}$$

The second term here turns out to be $\mathcal{O}(\lambda^{-N})$ for any N since the phase function in the oscillatory integral has no critical points. To be fully rigorous about this, we should reparametrize this as a surface integral in a tubular neighborhood of the equator on the sphere, e.g. using spherical coordinates, but we will wave our hands over this technicality.

We are now reduced to computing the asymptotics of $I(\lambda)$ using the stationary phase formula. Again (1.5) shows that the only critical point for $I(\lambda)$ is at the origin. Moreover,

$$\nabla\phi(0) = \frac{-I_{n-1}}{\sqrt{1-|y|^2}}\bigg|_{y=0} + \frac{yy^T}{(\sqrt{1-|y|^2})^3}\bigg|_{y=0} = -I_{n-1}$$

where I_{n-1} is the identity matrix in $n-1$ variables. Hence the phase is nondegenerate at the critical points. The stationary phase formula now gives us that

$$I(\lambda) = (2\pi)^{\frac{n-1}{2}}\lambda^{-\frac{n-1}{2}}|\det\nabla^2\phi(0)|^{-\frac{1}{2}}e^{\frac{i\pi}{4}\mathrm{sgn}\nabla^2\phi(0)}e^{i\lambda\phi(0)}\frac{\psi(0)}{\sqrt{1-0^2}} + \mathcal{O}(\lambda^{-\frac{n+1}{2}}).$$

The very last factor in the first term is clearly 1, but we have included it for emphasis as it is the result of evaluating the amplitude at $y=0$. Since $|\det\nabla^2\phi(0)| = 1$, $\mathrm{sgn}\nabla^2\phi(0) = -(n-1)$, and $\phi(0) = 1$, we now have

$$I(\lambda) = (2\pi)^{\frac{n-1}{2}}\lambda^{-\frac{n-1}{2}}e^{i\lambda - \frac{i\pi}{4}(n-1)} + \mathcal{O}(\lambda^{-\frac{n+1}{2}}).$$

Taking twice the real part of this expression and recalling that $|\xi| = \lambda$ then results in (1.2). □

Corollary 1.2. *The function $\widehat{u}(\xi)$ in (1.1) satisfies*

$$\widehat{u}(\xi) = Re\left(e^{i|\xi|}b(|\xi|)\right)$$

where $b : [0,\infty) \to \mathbb{C}$ is a smooth function satisfying for $j \geq 0$

$$\left|\frac{d^j}{d\lambda^j}b(\lambda)\right| \leq C_j(1+\lambda)^{-\frac{n-1}{2}-j}. \tag{1.8}$$

Proof. First observe that since $\widehat{u}$ is C^∞ and radial, then when $\lambda = |\xi| \leq 1$, (1.8) follows since each derivative is bounded on the closed unit ball about the origin. Hence we only need to treat the case $\lambda = |\xi| > 1$, which is assumed in the following.

In (1.6), (1.7), we used that $\widehat{u}$ is radial to see that with $|\xi| = \lambda$,

$$\widehat{u}(\xi) = 2\mathrm{Re}(I(\lambda)) + E(\lambda), \qquad E(\lambda) = \sum_{\pm}\int_{B(0,1)} e^{\mp i\lambda\sqrt{1-|y|^2}}\frac{(1-\psi)(y)}{\sqrt{1-|y|^2}}\,dy$$

so that $E(\lambda)$ is the contribution of the second term in (1.7). As before, the $E(\lambda)$ contribution can be reparameterized so that it is an oscillatory integral with no critical points, and hence satisfies

stronger bounds

$$\left|\frac{d^j}{d\lambda^j}E(\lambda)\right| \le C_{j,N}\lambda^{-N} \tag{1.9}$$

for any $N \ge 0$. Since $E(\lambda)$ is real valued,

$$\widehat{u}(\xi) = 2\mathrm{Re}(I(\lambda) + E(\lambda) = \mathrm{Re}(2I(\lambda) + E(\lambda)).$$

We therefore take

$$b(\lambda) = 2e^{-i\lambda}I(\lambda) + e^{-i\lambda}E(\lambda).$$

Given (1.9), and the product rule, the second term here satisfies the bounds in (1.8). Exercise 3b in Assignment 5 then yields the desired bounds on $\frac{d^j}{d\lambda^j}(2e^{-i\lambda}I(\lambda))$. □

Theorem 1.3. *Let $B = B(0,1) \subset \mathbb{R}^n$ be the unit ball in $\mathbb{R}^n$. There is a constant C such that*

$$|\widehat{\mathbf{1}_B}(\xi)| \le C(1+|\xi|)^{-\frac{n+1}{2}} \quad \textit{for all } \xi \in \mathbb{R}^n. \tag{1.10}$$

Proof. The case $n = 1$ is straightforward and left to the reader, so assume $n \ge 2$ in what follows. Since $\mathbf{1}_B \in L^1(\mathbb{R}^n)$, its Fourier transform is bounded, so the bound (1.10) is satisfied when $|\xi| \le 1$. It thus suffices to show that $|\widehat{\mathbf{1}_B}(\xi)| \le C|\xi|^{-\frac{n+1}{2}}$ for $|\xi| > 1$.

In what follows C is a constant that may change from line to line. Observe that

$$\widehat{\mathbf{1}_B}(\xi) = \int_B e^{-iy\cdot\xi}\,dy = \int_0^1 \left(\int_{\mathbb{S}^{n-1}} e^{-irx\cdot\xi}d\sigma(x)\right) r^{n-1}dr$$

where we have used polar coordinates $y = rx$ with $x \in \mathbb{S}^{n-1}$. The integral in parentheses is $\widehat{u}(r\xi)$ where $\widehat{u}$ is defined in (1.1). Corollary 1.2 then shows that $\widehat{\mathbf{1}_B}(\xi)$ is the real part of the following integral

$$\begin{aligned}
\int_0^1 e^{ir|\xi|}b(r|\xi|)r^{n-1}dr &= \frac{1}{i|\xi|}\int_0^1 \partial_r\left(e^{ir|\xi|}\right)b(r|\xi|)r^{n-1}dr \\
&= \frac{1}{i|\xi|}e^{ir|\xi|}b(r|\xi|)r^{n-1}\Big|_{r=0}^{r=1} - \frac{1}{i|\xi|}\int_0^1 e^{ir|\xi|}\partial_r\left(b(r|\xi|)r^{n-1}\right)dr \\
&= \frac{e^{i|\xi|}b(|\xi|)}{i|\xi|} - \frac{1}{i|\xi|}\int_0^1 e^{ir|\xi|}\left(b'(r|\xi|)|\xi|r^{n-1} + (n-1)b(r|\xi|)r^{n-2}\right)dr.
\end{aligned} \tag{1.11}$$

Here we have used that $\frac{1}{i|\xi|}e^{ir|\xi|}b(r|\xi|)r^{n-1}|_{r=0} = 0$. The first term here is bounded using (1.8):

$$\left|\frac{e^{i|\xi|}b(|\xi|)}{i|\xi|}\right| = |\xi|^{-1}|b(|\xi|)| \le C|\xi|^{-1}|\xi|^{-\frac{n-1}{2}} = C|\xi|^{-\frac{n+1}{2}}$$

We then have

$$\begin{aligned}
&\left|\frac{1}{i|\xi|}\int_0^1 e^{ir|\xi|}\left(b'(r|\xi|)|\xi|r^{n-1} + (n-1)b(r|\xi|)r^{n-2}\right)dr\right| \\
&\qquad \le \frac{1}{|\xi|}\int_0^1 |b'(r|\xi|)|\xi|r^{n-1} + (n-1)|b(r|\xi|)|r^{n-2}\,dr \\
&\qquad \le C\frac{1}{|\xi|}\int_0^1 (r|\xi|)^{-\frac{n-1}{2}-1}|\xi|r^{n-1} + (r|\xi|)^{-\frac{n-1}{2}}r^{n-2}\,dr = \frac{2C}{|\xi|^{\frac{n+1}{2}}}\int_0^1 r^{\frac{n-3}{2}}dr,
\end{aligned}$$

where we have used (1.8) in the second inequality and that $(1+r|\xi|)^{-1} \le (r|\xi|)^{-1}$. The integral on the very right here is always finite since our assumption $n \ge 2$ means that $(n-3)/2 > -1$. Hence the second term in (1.11) is bounded by $C|\xi|^{-\frac{n+1}{2}}$ as well. □

Theorem 1.4. *Suppose $n \geq 2$ and that $N(\lambda) = \#\{m \in \mathbb{Z}^n : |m| < \lambda\}$ is the number of integer lattice points in the ball of radius λ about the origin. Then*

$$N(\lambda) = Vol(B)\lambda^n + R(\lambda), \text{ where } |R(\lambda)| \leq C\lambda^{n-2+\frac{2}{n+1}}. \tag{1.12}$$

In other words, $N(\lambda) = Vol(B)\lambda^n + \mathcal{O}(\lambda^{n-2+\frac{2}{n+1}})$.

Note that since $n - 2 + \frac{2}{n+1} < n - 1 < n$, the remainder in (1.12) is indeed of lower order.

Proof. Recall the Poisson summation formula

$$\sum_{m\in\mathbb{Z}^n} g(2\pi m) = (2\pi)^{-n} \sum_{m\in\mathbb{Z}^n} \widehat{g}(m).$$

We will instead use the following rescaled version of this identity:

$$\sum_{m\in\mathbb{Z}^n} f(m) = \sum_{m\in\mathbb{Z}^n} \widehat{f}(2\pi m) \tag{1.13}$$

Indeed, taking $g(x) = f(x/2\pi)$, we have that $\widehat{g}(\xi) = (2\pi)^n \widehat{f}(2\pi\xi)$, so (1.13) does indeed follow from the first version of the Poisson summation formula.

Let $\chi_\eta = \mathbf{1}_{B(0,\eta)}$ denote the characteristic function of the ball of radius η. Note that

$$N(\lambda) = \sum_{m\in\mathbb{Z}^n} \chi_\lambda(m). \tag{1.14}$$

Hence ideally, we would apply the Poisson summation formula (1.13) with $f = \chi_\eta$ and $\eta = \lambda$, but we do not have enough decay of $\widehat{\chi_\eta}$ to satisfy the hypotheses of the theorem, which require that $f, \widehat{f}$ are $\mathcal{O}(|\xi|^{-n-\varepsilon})$ as $|\xi| \to \infty$. Indeed, $\chi_\eta(x) = \chi_1(x/\eta)$ (since χ_1 is just the characteristic function of the unit ball) and hence

$$|\widehat{\chi}_\eta(\xi)| = \eta^n |\widehat{\chi_1}(\eta\xi)| \leq C\eta^n (1 + \eta|\xi|)^{-\frac{n+1}{2}}, \tag{1.15}$$

showing that the decay at infinity is insufficient to apply (1.13). We therefore have to regularize χ_η, by convolving with an approximation to the identity.

Let $\beta \in C_c^\infty(B(0,1))$ be a radial, nonnegative function satisfying $\int \beta(x)dx = 1$ and set $\beta_\varepsilon(x) = \varepsilon^{-n}\beta(x/\varepsilon)$. We then define $\tilde{\chi}_\eta = \chi_\eta * \beta_\varepsilon$. In what follows, we will treat ε as fixed, but all the constants in the inequalities which follow can be taken independently of ε as long as $0 < \varepsilon < 1$. In particular, we will use that for η sufficiently large, $\eta \approx \eta \pm \varepsilon$.

As in the proof of the C^∞ Urysohn lemma, we have that $0 \leq \tilde{\chi}_\eta(x) \leq 1$ for each x and that

$$\tilde{\chi}_\eta(x) = \begin{cases} 1, & |x| \leq \eta - \varepsilon \\ 0, & |x| \geq \eta + \varepsilon. \end{cases}$$

Indeed, since $\operatorname{supp}(\beta_\varepsilon(x - \cdot)) \subset \{y : |y| < \varepsilon\}$,

$$\tilde{\chi}_\eta(x) = \int_{B(0,\eta)} \beta_\varepsilon(x - y)\, dx = \int_{B(0,\eta)\cap B(x,\varepsilon)} \beta_\varepsilon(x - y)\, dx,$$

so if $|x| \leq \eta - \varepsilon$, $B(0,\eta) \cap B(x,\varepsilon) = B(x,\varepsilon)$ so that the integral is 1, and if $|x| \geq \eta + \varepsilon$, $B(0,\eta)$ and $B(x,\varepsilon)$ are disjoint, so the integral vanishes. Moreover, for any x, we have $0 \leq \tilde{\chi}_\eta(x) \leq 1$ since $\beta \geq 0$ and $\int \beta_\varepsilon = 1$.

We are now interested in $\tilde{\chi}_{\lambda\pm\varepsilon}$. The above shows that

$$\operatorname{supp}(\tilde{\chi}_{\lambda-\varepsilon}) \subset B(0, (\lambda - \varepsilon) + \varepsilon) = B(0,\lambda) = \{x : \chi_\lambda(x) = 1\},$$
$$\{x : \tilde{\chi}_{\lambda+\varepsilon}(x) = 1\} \supset \overline{B(0, (\lambda - \varepsilon) + \varepsilon)} = \overline{B(0,\lambda)} = \operatorname{supp}(\chi_\lambda),$$

which implies the following pointwise inequalities for any $x \in \mathbb{R}^n$

$$\tilde{\chi}_{\lambda-\varepsilon}(x) \leq \chi_\lambda(x) \leq \tilde{\chi}_{\lambda+\varepsilon}(x). \tag{1.16}$$

Consequently, defining $\tilde{N}(\eta) = \sum_{m\in\mathbb{Z}^n} \tilde{\chi}_\eta(m)$ as an approximation to (1.14), we have that

$$\tilde{N}(\lambda-\varepsilon) \le N(\lambda) \le \tilde{N}(\lambda+\varepsilon). \tag{1.17}$$

Indeed, this is the result of taking m in (1.16) and summing over all $m \in \mathbb{Z}^n$.

The main idea is now to show that for η sufficiently large,

$$\tilde{N}(\eta) = \mathrm{Vol}(B)\eta^n + \tilde{R}(\eta), \text{ where } |\tilde{R}(\eta)| \le C\eta^{\frac{n-1}{2}}\varepsilon^{-\frac{n-1}{2}}. \tag{1.18}$$

As soon as we show this, then (1.12) will follow. Indeed, since $\lambda \pm \varepsilon \approx \lambda$, we have that

$$\tilde{N}(\lambda\pm\varepsilon) = \mathrm{Vol}(B)(\lambda\pm\varepsilon)^n + \tilde{R}(\lambda\pm\varepsilon) = \mathrm{Vol}(B)\lambda^n + \mathcal{O}(\lambda^{n-1}\varepsilon) + \mathcal{O}(\lambda^{\frac{n-1}{2}}\varepsilon^{-\frac{n-1}{2}}). \tag{1.19}$$

It is verified that taking $\varepsilon = \lambda^{-\frac{n-1}{n+1}}$ then ensures that the lower order terms are the same:

$$\lambda^{n-1}\varepsilon = \lambda^{\frac{n-1}{2}}\varepsilon^{-\frac{n-1}{2}} = \lambda^{n-2+\frac{2}{n+1}}.$$

Hence $\tilde{N}(\lambda\pm\varepsilon) = \mathrm{Vol}(B)\lambda^n + R_\pm(\lambda)$, where $R_\pm(\lambda) = \mathcal{O}(\lambda^{n-2+\frac{2}{n+1}})$, which shows that

$$R_-(\lambda) = \tilde{N}(\lambda-\varepsilon) - \mathrm{Vol}(B)\lambda^n \le N(\lambda) - \mathrm{Vol}(B)\lambda^n \le \tilde{N}(\lambda+\varepsilon) - \mathrm{Vol}(B)\lambda^n = R_+(\lambda),$$

that is, $|N(\lambda) - \mathrm{Vol}(B)\lambda^n| \le \max(R_-(\lambda), R_+(\lambda)) \le C\lambda^{n-2+\frac{2}{n+1}}$, concluding the proof of (1.12). Before proceeding with (1.18), it is interesting to note that $\varepsilon = \lambda^{-\frac{n-1}{n+1}}$ is indeed the optimal choice given the two error terms in (1.19). Indeed, we ultimately want the remainder in theorem to be as small as possible, so we want to take ε as small as possible. But the second error term in (1.19) is an inverse power of ε, penalizing us in taking ε small. The optimal choice thus results from taking ε to satisfy $\lambda^{n-1}\varepsilon = \lambda^{\frac{n-1}{2}}\varepsilon^{-\frac{n-1}{2}}$, which is the same as taking $\varepsilon = \lambda^{-\frac{n-1}{n+1}}$.

We are now left to show (1.18). Applying (1.13) we have that

$$\tilde{N}(\eta) = \sum_{m\in\mathbb{Z}^n} \tilde{\chi}_\eta(m) = \sum_{m\in\mathbb{Z}^n} \widehat{\tilde{\chi}_\eta}(2\pi m) = \sum_{m\in\mathbb{Z}^n} \widehat{\chi_\eta}(2\pi m)\widehat{\beta}_\varepsilon(2\pi m) = \sum_{m\in\mathbb{Z}^n} \widehat{\chi_\eta}(2\pi m)\widehat{\beta}(2\pi\varepsilon m),$$

using the usual formula for the Fourier transform of a convolution and that $\widehat{\beta}_\varepsilon(\xi) = \widehat{\beta}(\varepsilon\xi)$ using the usual dilation formula. Note that the $m = 0$ term in the sum here is

$$\widehat{\chi_\eta}(0)\widehat{\beta}(0) = \mathrm{Vol}(B)\eta^n.$$

Indeed, the value of the Fourier transform at the origin is simply the mean of the function, so $\widehat{\chi_\eta}(0) = \int \mathbf{1}_{B(0,\eta)} = \mathrm{Vol}(B)\eta^n$ and $\widehat{\beta}(0) = \int \beta = 1$.

It remains to show that

$$\left|\sum_{m\neq 0} \widehat{\chi_\eta}(2\pi m)\widehat{\beta}(2\pi\varepsilon m)\right| \le C\eta^{\frac{n-1}{2}}\varepsilon^{-\frac{n-1}{2}}.$$

Treating the sum as a Riemann sum for the integral $\int_{|\xi|\ge 1/2} \widehat{\chi_\eta}(2\pi\xi)\widehat{\beta}(2\pi\varepsilon\xi)$, we have

$$\begin{aligned}
\left|\sum_{m\neq 0} \widehat{\chi_\eta}(2\pi m)\widehat{\beta}(2\pi\varepsilon m)\right| &\approx \left|\int_{|\xi|\ge 1/2} \widehat{\chi_\eta}(2\pi\xi)\widehat{\beta}(2\pi\varepsilon\xi)d\xi\right| \\
&\le C\eta^n \int_{|\xi|\ge 1/2} (1+\eta|2\pi\xi|)^{-\frac{n+1}{2}}(1+|2\pi\varepsilon\xi|)^{-(n+1)}d\xi \\
&\approx C\eta^n \int_{|\xi|\ge 1/2} (1+\eta|\xi|)^{-\frac{n+1}{2}}(1+\varepsilon|\xi|)^{-(n+1)}d\xi
\end{aligned}$$

where we have used (1.15) and that $\widehat{\beta} \in \mathcal{S}(\mathbb{R}^n)$ and hence rapidly decreasing[1]. We now estimate the very last integral. When $|\xi| \le \varepsilon^{-1}$, the second factor in the integrand does not yield any appreciable

[1]Getting rid of the 2π in the last expression isn't really needed, it just makes the computations a little neater.

decay, so we will split the integral into regions where $1/2 \leq |\xi| \leq \varepsilon^{-1}$ and $|\xi| > \varepsilon^{-1}$. Taking polar coordinates in the integral we are reduced to estimating

$$\eta^n \int_{r\geq 1/2} (1+\eta r)^{-\frac{n+1}{2}} (1+\varepsilon r)^{-(n+1)} r^{n-1} dr$$
$$\leq \eta^n \int_{1/2}^{\varepsilon^{-1}} (\eta r)^{-\frac{n+1}{2}} r^{n-1} dr + \eta^n \int_{\varepsilon^{-1}}^{\infty} (\eta r)^{-\frac{n+1}{2}} (\varepsilon r)^{-(n+1)} r^{n-1} dr$$

The first integral can be estimated as

$$\eta^n \int_{1/2}^{\varepsilon^{-1}} (\eta r)^{-\frac{n+1}{2}} r^{n-1} dr = \eta^{\frac{n-1}{2}} \int_{1/2}^{\varepsilon^{-1}} r^{\frac{n-3}{2}} dr = \eta^{\frac{n-1}{2}} \frac{2r^{\frac{n-1}{2}}}{n-1}\Bigg|_{r=1/2}^{r=\varepsilon^{-1}} \leq \frac{2}{n-1} \eta^{\frac{n-1}{2}} \varepsilon^{-\frac{n-1}{2}}.$$

The second integral can be estimated as

$$\eta^n \int_{\varepsilon^{-1}}^{\infty} (\eta r)^{-\frac{n+1}{2}} (\varepsilon r)^{-(n+1)} r^{n-1} dr = \eta^{\frac{n-1}{2}} \varepsilon^{-(n+1)} \int_{\varepsilon^{-1}}^{\infty} r^{-\frac{n+5}{2}} dr = \eta^{\frac{n-1}{2}} \varepsilon^{-(n+1)} \left[-\frac{2}{n+3} r^{-\frac{n+3}{2}} \right]_{\varepsilon^{-1}}^{\infty}$$

and this last expression is $\mathcal{O}(\eta^{\frac{n-1}{2}} \varepsilon^{-(n+1)} \varepsilon^{\frac{n+3}{2}}) = \mathcal{O}(\eta^{\frac{n-1}{2}} \varepsilon^{-\frac{n-1}{2}})$. $\square$