

Math 562, Spring 2019
Assignment 6, due Wednesday, April 24

Note: You should be able to solve these exercises using the methods discussed in class, text, and the handouts. Avoid appealing to topological principles developed outside this class.

Hand in solutions to the following exercises:

1. Suppose $U \subset \mathbb{C}$ is open and simply connected. Let $f_n : U \rightarrow \mathbb{C} \setminus \{0, 1\}$ be a sequence of holomorphic functions. Suppose there is $z_0 \in U$ such that $\lim_{n \rightarrow \infty} f_n(z_0) = 0$. Prove that f_n converges normally to 0.

Hint: It may be helpful to show that if the conclusion is false, there exists a subsequence $\{f_{n_j}\}_{j=1}^\infty$ which does not converge normally to 0, then derive a contradiction by taking a further subsequence. Just make sure to keep your (Hur-)wits about you.

2. (a) Suppose $U \subset \mathbb{C}$ is open and connected, and let $\gamma_0, \gamma_1 : [0, 1] \rightarrow U$ be curves in U , with a common initial point P and terminal point Q . Show that if γ_0, γ_1 are homotopic, through a fixed endpoint homotopy as in Definition 10.3.1, then $\oint_{\gamma_0} f = \oint_{\gamma_1} f$.

(b) Greene & Krantz, Chapter 11, Exercise 3.

Note: Part (a) is designed for you to be able to use the hint in (b).

3. Find all possible values of $\oint_\gamma \frac{dz}{1+z^2}$ where γ is a piecewise C^1 curve in \mathbb{C} not passing through $\pm i$.

Note: You may use the version of the residue theorem on p. 125 of the text. To that end, think of this as more of an exercise concerning the index function.

4. Suppose $U \subset \mathbb{C}$ is bounded, open, and connected, but *not* simply connected. Recall that a holomorphic function $f : U \rightarrow \mathbb{C}$ has a holomorphic antiderivative if there exists $F : U \rightarrow \mathbb{C}$ such that $F' \equiv f$ on U . Show there exist a function of the form $f(z) = (z - a)^{-1}$ which is holomorphic on U but does not possess a holomorphic antiderivative. Justify your answer.
5. Suppose $U \subset \mathbb{C}$ is open and simply connected. Let $f : U \rightarrow \mathbb{C}$ be a nonvanishing holomorphic function. Prove that there exists a sequence $\{p_n\}_{n=1}^\infty$ such that $p_n^2 \rightarrow f$ uniformly on compact sets.

Hint: The simply connected hypothesis should play a role in 2 stages of the proof.

Reading: §11.1-11.4 and §12.1. If you've had some training in topology, then §11.5 is a good section to read, though we will not do anything with it in class. We will only cover 12.1 (Runge's theorem) in Chapter 12.

On your own: Greene & Krantz: Chapter 11, Exercises 12, 13. Also, the following exercises:

1. A connected set $U \subset \mathbb{C}$ is said to be *star-shaped* if there exists $z_0 \in U$ such that for any $z \in U$, the line segment joining z and z_0 is contained in U . Convex sets are star-shaped, but not conversely: just take z_0 to be the center of any region shaped like a hand drawn star (\star) to get an example of a set which is star-shaped but not convex.

Prove that any star-shaped U is simply connected.