Math 562, Spring 2019
Assignment 5, due Wednesday, April 10

Hand in solutions to the following exercises:

1. Let $U \subset \mathbb{C}$ be open and let $f = u + iv : U \to \mathbb{C}$ be continuously differentiable, with $u = \operatorname{Re} f$, $v = \operatorname{Im} f$ as usual. Let $\bar{U} = \{\bar{z} : z \in U\}$.
   (a) Show that the following are equivalent:
      i. $z \mapsto f(\bar{z})$ is holomorphic on $\bar{U}$.
      ii. $z \mapsto \overline{f(z)}$ is holomorphic on U.
      iii. $\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$ on U.
      iv. $\frac{\partial f}{\partial z} = 0$ on U.

      We say that f is *conjugate holomorphic* if it satisfies any of these 4 conditions.
   (b) Recall from §2.2 of Greene & Krantz that if $|w| = 1$, the directional derivative of f in the direction of w is defined as $D_w f(P) = \lim_{t \to 0} \frac{f(P+tw)-f(P)}{t}$. Prove that if f is conjugate holomorphic and $\frac{\partial f}{\partial \bar{z}}$ does not vanish at P, then it reverses angles in the following sense: if $|w_1|, |w_2| = 1$ and $e^{i\theta} w_1 = w_2$, then the directional derivatives of f satisfy $e^{-i\theta} D_{w_1} f(P) = D_{w_2} f(P)$.

2. Explain the fallacy in this apparent counterexample to the little Picard theorem:

   "Let $f(z) = \exp(\exp z)$, that is, f is the composition of $\exp z$ with itself. This is entire as it is the composition of 2 entire functions. Since $\exp z$ never assumes the value 0, neither does f since $f(\mathbb{C}) \subset \exp(\mathbb{C})$. Moreover, $f(z) = \exp(\exp z) \neq e^0 = 1$ for any $z \in \mathbb{C}$. Hence f omits both 0 and 1 from its range, but is nonconstant, showing the little Picard theorem is false."

3. Let f be a nonconstant entire function such that $f(1 + z) = 1 + f(z)$ for all $z \in \mathbb{C}$. Determine the range of f, that is, the set $f(\mathbb{C})$. Justify your answer.

4. Suppose $f : \mathbb{C} \to \widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ is a meromorphic function, where as usual, we take $f(P) = \infty$ if f has a pole at P. Show that if $\widehat{\mathbb{C}} \setminus f(\mathbb{C})$ contains at least three distinct elements, then f is constant.

   Hint: Use linear fractional transformations to reduce to the case where $\infty \notin f(\mathbb{C})$.

5. Use the *big* Picard theorem to show that an entire function which is not a polynomial takes on every complex value but one infinitely often.

Reading: Read through the handout on covering maps and the Picard theorem.

On your own: Greene & Krantz: Chapter 6, Exercise 30 (most of this was done in class). Also, the following exercises:

1. Prove that if f is a nonconstant entire function and $b^2 \neq 4ac$, then the function $g(z) = af^2(z) + bf(z) + c$ has a zero.

2. Show that if f is conjugate holomorphic, then its real and imaginary parts are harmonic.
3. Show that if w is harmonic, f is conjugate holomorphic, and $w \circ f$ is defined, then the composition is also harmonic.