Math 562, Spring 2019
Assignment 4, due Wednesday, March 27

Hand in solutions to the following exercises:

1. Suppose $0 < \lambda_1 < \lambda_2 < \cdots$ with $\lim_{n\to\infty} \lambda_n = \infty$. Prove that there exists a *bounded* holomorphic function on the upper half plane $\{z : \operatorname{Im} z > 0\}$. with simple zeros at $i + \lambda_n$ if and only if $\sum_{n=1}^{\infty} \lambda_n^{-2}$ is convergent.

   Hint: Begin by transferring the problem to the unit disc using the Cayley transform $w \mapsto \frac{w-i}{w+i}$. Don't forget about the *limit comparison test* from calculus: if $a_n, b_n > 0$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ with $0 < L < \infty$, then $\sum a_n$ converges if and only if $\sum b_n$ converges.

2. (a) Suppose f is an entire function of finite order $\lambda(f)$, $k \geq 0$ is an integer, and $g(z) = f(z^k)$. Prove that g is of finite order given by $\lambda(g) = k\lambda(f)$.

   (b) Prove that if f is an even entire function of finite order $\lambda(f)$ and $g(z) = f(\sqrt{z})$, then g is an entire function of order $\lambda(g) = \lambda(f)/2$. In particular, show that $\cos(\sqrt{z})$ is of order $1/2$.

   Note: Don't forget that the order λ of an entire function f is characterized by 2 properties: for any $\epsilon > 0$, $|f(z)| \leq \exp(|z|^{\lambda+\epsilon})$ for z sufficiently large and moreover $|f(z)| \not\leq \exp(|z|^{\lambda-\epsilon})$ as $|z| \to \infty$. For the latter, it is necessary and sufficient that there exists a sequence $|z_n| \to \infty$ such that $|f(z_n)| > \exp(|z_n|^{\lambda-\epsilon})$.

3. Greene & Krantz, Chapter 10, Exercise 1.

4. Let $U_0 = D(1,1)$ and let f_0 be the restriction of $\log z$ to U_0 where $\log z$ is defined by $\log(re^{i\theta}) = \log r + i\theta$ with $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. For $n \in \mathbb{Z}$, let $\gamma(t) = \exp(2\pi i n t)$, $0 \leq t \leq 1$. Show that any analytic continuation $\{(f_t, U_t)\}_{0\leq t\leq 1}$ of (f_0, U_0) along $\gamma(t)$ satisfies at $t = 1$, $f_1 = f_0 + 2\pi i n$ on $U_0 \cap U_1$ (which must contain 1 since $\gamma(0) = \gamma(1) = 1$).

On your own: Greene & Krantz: Chapter 10, Exercises 2, 5, 6, 10. Also, the following exercises:

1. Prove that if f is an entire function of order λ, then f' also has order λ.

2. Consider the circumstance in Exercise 2b above: if f is an even entire function, how exactly is $g(z) = f(\sqrt{z})$ defined so that it also gives an entire function?

3. The collection $\{U_0, U_1, \ldots, U_n\}$ of open disks is called a *chain of disks* if $U_{j-1} \cap U_j \neq \emptyset$ for $j = 1, \ldots, n$. If $(f_0, U_0), \ldots, (f_n, U_n)$ are function elements such that $f_j = f_{j-1}$ on $U_{j-1} \cap U_j$, then this collection is called an *analytic continuation along a chain of disks.*

   (a) Let $(f_0, U_0), \ldots, (f_n, U_n)$ be an analytic continuation along a chain of disks and let P, Q be the centers of U_0, U_n respectively. Show that there is a curve γ from P to Q and an analytic continuation $\{(g_t, V_t)\}_{0\leq t\leq 1}$ along γ such that the image of γ is contained in $\cup_{j=1}^{n} U_j$ with $g_0 = f_0$ on $U_0 \cap V_0$ and $g_1 = f_n$ on $U_n \cap V_1$.

(b) Conversely, let $\{(g_t, V_t)\}_{0\leq t\leq 1}$ be an analytic continuation along a curve γ from P to Q. Show that there is an analytic continuation along a chain of disks $(f_0, U_0), \ldots, (f_n, U_n)$ such that the image of γ is contained in $\cup_{j=1}^{n} U_j$ with $g_0 = f_0$ on $U_0 \cap V_0$ and $g_1 = f_n$ on $U_n \cap V_1$.

Reading: Greene & Krantz, Chapter 10.