## Math 562, Spring 2019 <br> Assignment 3, due Wednesday, February 20

## Hand in solutions to the following exercises:

1. Let $U \subset \mathbb{C}$ be an open set with $0 \notin U$ and define $\tilde{U}:=\{z: 1 / \bar{z} \in U\}$. Given $f: U \rightarrow \mathbb{C}$ define $\tilde{f}: \tilde{U} \rightarrow \mathbb{C}$ by $\tilde{f}(z):=\overline{f(1 / \bar{z})}$
(a) Prove that if $f$ is holomorphic on $U$, then $\tilde{f}$ is holomorphic on $\tilde{U}$.
(b) Suppose that $U=\tilde{U}$, that is, $U$ is unchanged under the operation $z \mapsto 1 / \bar{z}$. Suppose further that $U \cap\{z:|z|=1\} \neq \emptyset$ and that $f$ is real-valued on this set. Prove that $f \equiv \tilde{f}$.
2. (a) Greene \& Krantz, Chapter 8, Exercise 1.
(b) Greene \& Krantz, Chapter 8, Exercise 2.
3. Prove that

$$
\sin (\pi z)=\pi z \prod_{n=1}^{\infty}\left(1-\frac{z^{2}}{n^{2}}\right)
$$

Note: You may assume the following identity, as it is a consequence of the residue theory from Math 561 (cf. The "On your own" portion of Assignment 11):

$$
\pi \cot (\pi z)=\lim _{N \rightarrow \infty} \sum_{n=-N}^{N} \frac{1}{n+z}=\frac{1}{z}+\sum_{n=1}^{\infty} \frac{2 z}{z^{2}-n^{2}}, \quad z \notin \mathbb{Z}
$$

4. Show that the equation $e^{z}=z$ has infinitely many solutions in $\mathbb{C}$.

Hint: Try a proof by contradiction. What does the Weierstrass factorization theorem tell you about $f(z):=e^{z}-z$ if there are finitely many zeros?
5. Greene \& Krantz, Chapter 8, Exercise 20.

On your own: Greene \& Krantz: Chapter 8, Exercises 4, 6, 8. Also, the following exercises:

1. Derive the identity for $\pi \cot (\pi z)$ in $\# 3$ above if you've forgotten how.
2. Let $\left\{a_{n}\right\}_{n=1}^{\infty},\left\{\alpha_{n}\right\}_{n=1}^{\infty}$ be complex sequences. Using the Weierstrass factorization and Mittag-Leffler theorems, prove that if $\lim _{n \rightarrow \infty}\left|a_{n}\right|=\infty$ with all $a_{n}$ distinct, then there exists an entire function $f$ such that $f\left(a_{n}\right)=\alpha_{n}$.
Note: Test yourself and see if you can find a solution without peeking at Theorem 8.3.8. First find an entire function $g$ with $g\left(a_{n}\right)=0, g^{\prime}\left(a_{n}\right) \neq 0$. Next find a meromorphic function $h$ with simple poles and well-chosen residues at each $a_{n}$ so that $f=g h$ gives the desired function.

Reading: Greene \& Krantz, Chapter 8, start Chapter 9.

