# Math 562, Spring 2019
# Assignment 2, due Wednesday, February 6

**Hand in solutions to the following exercises**:

1. Greene & Krantz, Chapter 7, Exercise 19.

2. Find a function $u(z)$ harmonic on $U = \{z : |z| < 2, |z-1| > 1\}$ such that $u(z) = 1$ on the outer boundary $\{z : |z| = 2, z \neq 2\}$ and $u(z) = 3$ on the inner boundary $\{z : |z-1| = 1, z \neq 2\}$. As always, show your work and justify your answer.

   Hint: Map $U$ to a simpler domain by a Möbius transformation.

3. Let $D = D(0,1)$. Recall from the bottom of p. 213 in Greene & Krantz that when $a = re^{i\theta} \in D$, the Poisson kernel $\frac{1}{2\pi} \cdot \frac{1-|a|^2}{|a-e^{i\psi}|^2}$ can be rewritten as $P_r(\theta - \psi)$ where

$$P_r(\phi) = \frac{1}{2\pi} \cdot \frac{1-r^2}{1-2r\cos\phi + r^2}.$$

   Note that the factor of $\frac{1}{2\pi}$ is part of the definition of $P_r$

   (a) Prove the following identities for $0 \leq r < 1$:

$$2\pi P_r(\phi) = \mathrm{Re}\left(\frac{1+re^{i\phi}}{1-re^{i\phi}}\right) \qquad \text{and} \qquad 2\pi P_r(\phi) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\phi}$$

   (b) Greene & Krantz, Chapter 7, Exercise 32.

   (c) Suppose that $f : \overline{D} \to \mathbb{C}$ is a continuous function such that both $\mathrm{Re}\, f$ and $\mathrm{Im}\, f$ are harmonic on $D$. Show that $f$ is holomorphic on $D$ if and only if

$$\int_0^{2\pi} f(e^{i\phi}) e^{in\phi}\, d\phi = 0 \qquad \text{for all } n = 1, 2, 3, \ldots.$$

   Note: By summing over real and imaginary parts, it is not hard to see that $f(re^{i\theta}) = \int_0^{2\pi} f(e^{i\psi}) P_r(\theta - \psi) d\psi$ for $r < 1$.

4. Find all entire functions $f(z)$ which are analytic and bounded on the upper half plane $U = \{z : \mathrm{Im}\, z > 0\}$, continuous on $\overline{U}$, and real-valued on $\mathrm{Im}\, z = 0$ (the real axis).

**On your own**: Greene & Krantz: Ch. 1, Exercise 34[1]; Ch. 7, Exercise 1[2], 2[3], 4, 10, 23[4]. Also, the following exercises (see the next page):

[1] Yes, a Chapter 1 exercise! It's not hard, but is significant to revisit given the material in Ch. 7. In particular, how does this show that $\overline{F(\bar{z})}$ is holomorphic on $\{z : \bar{z} \in U\}$ whenever $F$ is holomorphic on $U$?

[2] Why is this result a consequence of Exercise 1 above (a.k.a. Exercise 19 in the text)?

[3] This may give insight to Exercise 23 in the text.

[4] This one isn't hard given Lemma 7.3.2, but it is important as it is a property that we will use later on.

1. (a) Show that if $u$ is a harmonic function in a holomorphically simply connected domain $U \subset \mathbb{C}$, then $u(z) = \log |f(z)|$ for some nowhere vanishing holomorphic function $f$ on $U$.

   (b) Find a harmonic conjugate of $u(x, y) = \frac{1}{2} \log(x^2 + y^2)$ on the domain $\mathbb{C} \setminus [0, +\infty)$.

**Reading:** Greene & Krantz, §7.1-7.6. We will not cover §7.7-7.8. Make sure to read through the book's proof of the maximum principle for harmonic functions (Theorem 7.2.1) as the trick there is commonly used for harmonic function problems.