Math 562, Spring 2019
Assignment 1, due Wednesday, January 30

Hand in solutions to the following exercises:

1. Find a linear fractional transformation that simultaneously maps the unit circle $|z| = 1$ and the line $y = x$ onto the coordinate axes $x = 0$ and $y = 0$ respectively. Make sure to justify your answer.

2. Suppose f is holomorphic on the right half plane $U = \{w \in \mathbb{C} : \operatorname{Re} w > 0\}$ and satisfies $|f(w)| < 1$ for all $w \in U$. Show that if $f(1) = 0$, then
$$|f(w)| \leq \left|\frac{w-1}{w+1}\right| \qquad \text{for all } w \in U.$$
Hint: Transfer the problem to the unit disc. Recall that the Cayley transform $z \mapsto i\frac{1-z}{1+z}$ maps the unit disc conformally to the *upper* half plane (cf. Ch. 1, Exercise 9).

3. Greene & Krantz, Chapter 6, Exercise 22.

   Hint: It may be helpful to revisit the last paragraph in the proof of Proposition 6.5.7.

4. (a) Let $D = D(0, 1)$ be the open unit disc and let P, Q be distinct points in D. Suppose $\phi : D(0, 1) \to D(0, 1)$ is a conformal map from the unit disc to itself which fixes P, Q, that is $\phi(P) = P$, $\phi(Q) = Q$. Prove that ϕ is the identity map.

   Note: Some algebraic tedium can be avoided by reducing to the case where $P = 0$, but this should be justified by appealing to the transformations in 6.2.2.

   (b) (Greene & Krantz, Chapter 6, Exercise 12 rewritten). Let $\Omega \subsetneq \mathbb{C}$ be a holomorphically simply connected domain and let P, Q be distinct points of Ω. Let ϕ_1 and ϕ_2 be conformal self-maps of Ω. If $\phi_1(P) = \phi_2(P)$ and $\phi_1(Q) = \phi_2(Q)$, then prove that $\phi_1 \equiv \phi_2$ (i.e. the functions are identically equal).

On your own: Greene & Krantz: Ch. 6, Exercise 20 (this is actually rather straightforward once you revisit Corollary 3.5.2). Also, the following exercises:

1. If $Tz = \frac{az+b}{cz+d}$, find z_2, z_3, z_4 such that Tz is equal to the cross-ratio (z, z_2, z_3, z_4)

2. Find the linear fractional transformation that maps $-1, 0, 1$ to $-i, 1, i$ respectively.

3. Given one conformal equivalence h from an open set $U \subset \mathbb{C}$ to $D = D(0, 1)$, can you describe all the others in terms of h?

4. (a) Prove (or find a proof) that any linear fractional transformation is the composition of translations, dilations, and the inversion map (i.e. $z \mapsto \frac{1}{z}$).

   (b) Greene & Krantz, Chapter 6, Exercise 31. The parenthetical comment is an important qualifier here, just think about what happens with inversion.

Reading: Greene & Krantz, finish Chapter 6, start Chapter 7. Review §3.5, §6.2, and the Schwarz lemma in preparation for the proof of the Riemann mapping theorem.