Math 561, Fall 2018
Assignment 6
Not Collected

On your own:

1. (a) Prove that the geometric series $\sum_{k=0}^{\infty} z^k$ has radius of convergence equal to 1 and converges to $\frac{1}{1-z}$ for $|z| < 1$.

   (b) Greene & Krantz, Chapter 3, Exercise #10.

2. Greene & Krantz, Chapter 3, Exercise #11, but replace the series in part (f) with
$$\sum_{k=0}^{\infty}(2+(-1)^k)^k z^k.$$

3. (a) Review the root and ratio tests from calculus/real analysis which give sufficient conditions for a series to converge absolutely. Why are they applicable to series whose terms are complex numbers?

   (b) As noted in Week 1, the function e^z can be defined by the power series
$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}.$$
   Prove that the radius of convergence of this power series is $R = \infty$. Conclude that e^z is an *entire* function, i.e. a function which is holomorphic on all of $\mathbb{C}$.

   (c) Define $\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$ and $\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$. Compute their power series, show that the radius of convergence is $R = \infty$ and conclude that they define entire functions. Then observe that these power series agree with the usual ones from calculus when $z \in \mathbb{R}$.

   (d) Greene & Krantz, Chapter 3, Exercise #15.

4. (a) Suppose f is analytic on $D(P, r)$ and unbounded on that disc in that there is no M satisfying $|f(z)| \le M$ for all $z \in D(P, r)$. Prove that the radius of convergence of power series for f about P is equal to r.

   (b) The function $\frac{1}{\cos z}$ has a power series expansion at $P = 0$. Without finding the series, show that its radius of convergence is $\frac{\pi}{2}$.
   Note: Exercise care here–showing that the radius of convergence is less than $\pi/2$ is not the same as showing that it is equal to $\pi/2$.

Reading: Greene & Krantz 3.2-3.3.