Math 561, Fall 2018
Assignment 4, due Wednesday, September 19

Hand in solutions to the following exercises:

1. Greene & Krantz, Chapter 2, Exercise #18.

2. Greene & Krantz, Chapter 2, Exercise #23.

3. Suppose that $F : \bar{D}(z_0, r_0) \to \mathbb{C}$ is holomorphic on the open disc $D(z_0, r_0)$ and continuous on the closed disc $\bar{D}(z_0, r_0)$. Let $\gamma : [0, 2\pi] \to \mathbb{C}$ be the curve $\gamma(t) = z_0 + r_0 e^{it}$. Prove that for any $z \in D(z_0, r_0)$,
$$F(z) = \frac{1}{2\pi i} \oint_\gamma \frac{F(\zeta)}{\zeta - z}\, d\zeta.$$
Note: Given $W \subset \mathbb{C}$, a function $f : W \to \mathbb{C}$ is said to be *uniformly continuous* if for any $\epsilon > 0$, there exists $\delta > 0$ such that $|f(w) - f(z)| < \epsilon$ whenever $w, z \in W$ are **any** two points satisfying $|w - z| < \delta$. In other words, f is continuous at all points $z \in W$, but the δ needed to ensure that $f(w), f(z)$ are ϵ-close can be taken **independently** of z. It is a standard property that given any closed, bounded set $W \subset \mathbb{C}$, then any continuous function f on W is in fact uniformly continuous. Use this property in the exercise to show that if $D = D(z_0, r_0)$, then for $z \in D$
$$\lim_{r \to r_0 -} \oint_{\partial D(z_0, r)} \frac{F(\zeta)}{\zeta - z}\, d\zeta = \oint_{\partial D(z_0, r_0)} \frac{F(\zeta)}{\zeta - z}\, d\zeta.$$

On your own: Greene & Krantz: Chapter 2, Exercises 19[1]. Also the following problems:

1. Formulate and prove a precise statement to the following effect: addition, multiplication, and division are continuous operations on $\mathbb{C}$.

2. Revisit Exercise #52 in Chapter 1, but this time disprove the existence of a holomorphic antiderivative by using line integrals.

3. Prove the power rule in the complex domain: $\frac{\partial}{\partial z} z^k = k z^{k-1}$ when $k \neq 0$ and $k \in \mathbb{Z}$. Begin with an inductive argument that treats $k \geq 1$, with the base case given by the identity $\frac{\partial}{\partial z} z = 1$ (which you may assume since it's in Chapter 1).

Reading: Greene & Krantz 2.3-2.6.

[1] You may assume differentiation under the integral sign is justified here.