Math 561, Fall 2018
Assignment 3, due Wednesday, September 12

Hand in solutions to the following exercises (from Chapters 1 and 2):

1. Greene & Krantz, Chapter 1, Exercise #52.

   Tip (in addition to the given hint): The antiderivative of $\frac{1}{z}$ "should" be $\log z$, the inverse function of e^z. But this raises the question of what exactly $\log z$ is! Think about how you might be able to define $\log(re^{i\theta})$ and show that it can be defined on a slit annulus, e.g. $\tilde{U} := U \setminus \{x \in \mathbb{R} : x < 0\}$, forcing the difference between this and any holomorphic antiderivative on U to be constant on $\tilde{U}$. The first exercise in Assignment 2 may be helpful to this end. Exercise #36 in Chapter 1 may be helpful to check that your $\log z$ does indeed give you a holomorphic antiderivative on $\tilde{U}$.

2. Complete these 2 closely related exercises:

   (a) Greene & Krantz, Chapter 2, Exercise #1.
   (b) Greene & Krantz, Chapter 2, Exercise #2.

3. Greene & Krantz, Chapter 2, Exercise #13.

   Note: Be careful with the chain rule part, either adapt the usual proof of the chain rule for functions of 1 real variable or appeal to Exercise #49 in Chapter 1 (along with any needed Theorems from Ch. 2).

4. Suppose that $U \subset \mathbb{C}$ is open and that $f : U \to \mathbb{C}$ is a C^1 function with $f = u + iv$ giving the real and imaginary parts. Recall the definition of directional derivative at a point $z \in U$ in the direction of $w \in \mathbb{C}$ from Theorem 2.2.3:

   $$D_w f(z) = \lim_{t \to 0} \frac{f(z + tw) - f(z)}{t}.$$

   Suppose that $e^{i\theta} D_{w_1} f(z) = D_{w_2} f(z)$ whenever $|w_1| = |w_2| = 1$ and $e^{i\theta} w_1 = w_2$. In other words, if w_2 is obtained by a rotation of w_1 through an angle of θ, then $D_{w_2} f(z)$ is obtained by a rotation of $D_{w_2} f(z)$ through an angle of θ. Prove that $\frac{\partial f}{\partial \bar{z}}$ vanishes at z (or equivalently the Cauchy-Riemann equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ are satisfied at the point z).

On your own: Greene & Krantz: Ch. 1, Exercise 51 and Ch. 2 Exercises 4, 5, 7, 10, 14. Also the following problems:

1. Identity (***) on p.32 of Greene & Krantz text states that

   $$f(\gamma(b)) - f(\gamma(a)) = \int_a^b \frac{\partial f}{\partial z}(\gamma(t)) \frac{d\gamma}{dt}(t)\, dt,$$

   giving a version of the fundamental theorem of calculus for holomorphic functions. The proof on p.31-32 starts with the left hand side of this identity, then manipulates this until the right hand side is obtained. As a means of familiarization, go the other way around with this derivation: start with the right hand side, then manipulate until you obtain the left hand side.

2. Prove Proposition 2.1.9 in the Greene & Krantz text. What happens if instead ϕ is a *decreasing* function?

Reading: Greene & Krantz 2.1-2.3.