Math 561, Fall 2018
Assignment 1, due Wednesday, August 29

1. Greene & Krantz, Chapter 1, Exercise #9.

2. Greene & Krantz, Chapter 1, Exercise #10.

   Note: For these first two exercises, part of the work is in showing that ranges of the functions ϕ, ψ are indeed contained in the upper half plane $U = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. Once this is done, you may want to proceed by constructing an inverse function using algebraic methods.

3. Greene & Krantz, Chapter 1, Exercise #12.

4. (a) Use Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ to derive *DeMoivre's formula*: $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$
   (b) Use Euler's formula to derive the identities
      i. $\sin(\theta \pm \psi) = \sin\theta\cos\psi \pm \cos\theta\sin\psi$
      ii. $\cos(\theta \pm \psi) = \cos\theta\cos\psi \mp \sin\theta\sin\psi$
   (c) Use induction on n to prove that $|\sin(n\theta)| \leq n|\sin\theta|$ for $n = 1, 2, 3 \ldots$.

5. Given $(x, y) \in \mathbb{R}^2$, define $M_{x,y}$ to be the 2×2 matrix

$$M_{x,y} = \begin{bmatrix} x & -y \\ y & x \end{bmatrix}.$$

   Let $\Omega = \{M_{x,y} : (x, y) \in \mathbb{R}^2\}$, that is, Ω is the set of 2×2 matrices of the form $M_{x,y}$. It's not hard to see that the mapping $\Phi : \mathbb{C} \to \Omega$ defined by $\Phi(z) = M_{x,y}$ with $x = \operatorname{Re} z$, $y = \operatorname{Im} z$ is a bijection and $\Phi(w + z) = \Phi(w) + \Phi(z)$ so Ω is closed under matrix addition. Moreover, $\Phi(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\Phi(i) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. If $z = x + iy$, $w = u + iv$ are in standard form, prove that Φ and Ω satisfy the following properties:

   (a) $\Phi(z \cdot w) = \Phi(z)\Phi(w)$, that is, $\Phi(z \cdot w)$ is the matrix product $M_{x,y}M_{u,v}$. As a corollary, observe that Ω is closed under matrix multiplication and that $M_{x,y}M_{u,v} = M_{u,v}M_{x,y}$.
   (b) $|z| = \det\Phi(z)$.
   (c) If $z \neq 0$, prove that $\Phi(1/z) = (M_{x,y})^{-1}$, that is, $\Phi(1/z)$ is the matrix inverse of $M_{x,y} = \Phi(z)$.
   (d) $\Phi(\bar{z}) = (M_{x,y})^T$, that is, $\Phi(\bar{z})$ is the transpose of $\Phi(z)$.
   (e) If $r = \sqrt{x^2 + y^2}$, then the matrix product $M_{x,y}M_{1/r,0}$ is a rotation matrix, that is, there exists θ such that

$$M_{x,y}M_{1/r,0} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

   Do this without appealing to part 5a.

   Given these results, $\mathbb{C}$ can be viewed as space of matrices Ω, though it is not typically advantageous to take this viewpoint. However, one exception is when considering the geometry of complex multiplication. To see this, fix $w_0 = u_0 + iv_0 \in \mathbb{C}$ and note that part 5a shows that

$$\begin{bmatrix} u_0x - v_0y & -(v_0x + u_0y) \\ v_0x + u_0y & u_0x - v_0y \end{bmatrix} = \begin{bmatrix} u_0 & -v_0 \\ v_0 & u_0 \end{bmatrix} \begin{bmatrix} x & -y \\ y & x \end{bmatrix}.$$

Hence the first column on the left hand side here determines the image of the point $(x, y) \in \mathbb{R}^2$ under $M_{u_0,v_0} = \Phi(w_0)$. This shows that the action of the linear map M_{u_0,v_0} on $\mathbb{R}^2$ is equivalent to taking the product $w_0 z$ for $z \in \mathbb{C}$.

6. Let (X, d) be a metric space and suppose $E \subset X$. Define the interior and closure of E as

$$E^\circ = \cup\{U : U \subset E \text{ and } U \text{ is open }\},$$
$$\bar{E} = \cap\{F : F \supset E \text{ and } F \text{ is closed }\}.$$

In other words, E° is the union of all open sets contained in E and $\bar{E}$ is the intersection of all closed sets containing E.

(a) Prove that $p \in E^\circ$ if and only if there exists $r > 0$ such that $B(p, r) \subset E$.

(b) Prove that $p \in \bar{E}$ if and only if $B(p, r) \cap E \neq \emptyset$ for every $r > 0$.

On your own (i.e. do not hand these in for a grade):

Greene & Krantz Ch. 1, Exercises 1-3 and 13-14 (as needed for review), 4, 5, 8, 22, 23 and the following problems:

1. Let X, Y be sets and let $f : X \to Y$ be a function. A *left inverse* for f is a function $g : Y \to X$ satisfying $(g \circ f)(x) = x$ for all $x \in X$. A *right inverse* for f is a function $h : Y \to X$ such that $f \circ h(y) = y$ for all $y \in Y$. Assuming the axiom of choice, prove the following:

(a) The function f has a left inverse if and only if it is injective.

(b) The function f has a right inverse if and only if it is surjective.

2. Let X, Y be sets and let $f : X \to Y$ be a function. Suppose $E, E_\alpha \subset Y$, $G \subset X$

(a) $f^{-1}(E^C) = [f^{-1}(E)]^C$

(b) $f(f^{-1}(E)) \subset E$

(c) $G \subset f^{-1}(f(G))$

(d) $f^{-1}(\cup_\alpha E_\alpha) = \cup_\alpha f^{-1}(E_\alpha)$

(e) $f^{-1}(\cap_\alpha E_\alpha) = \cap_\alpha f^{-1}(E_\alpha)$

Find examples of functions f such that equality in 2b and 2c fails to hold. Prove that equality in 2b holds whenever f is surjective and equality in 2c holds whenever f is injective.

3. Prove the following statements concerning open and closed sets in a metric space (X, d).

(a) X and $\emptyset$ are both open and closed.

(b) If $\{U_\alpha\}_{\alpha \in A}$ is an arbitrary collection of open sets, then $\cup_{\alpha \in A} U_\alpha$ is open.

(c) If $U_1, \ldots U_k$ is a finite collection of open sets, then $\cap_{j=1}^k U_j$ is open.

(d) If $\{F_\alpha\}_{\alpha \in A}$ is an arbitrary collection of closed sets, then $\cap_{\alpha \in A} F_\alpha$ is open.

(e) If $F_1, \ldots F_k$ is a finite collection of closed sets, then $\cup_{j=1}^k F_j$ is open.

(f) In general, the finite collection hypothesis in 3c and 3e is necessary.

Reading: Greene & Krantz §1.1-1.3.