Math 561, Fall 2018
Assignment 13, due Friday, December 7

Hand in solutions to the following exercises:

1. Greene & Krantz, Chapter 6, Exercise 14.

2. Let $U \subset \mathbb{C}$ be open with $P \in U$. Suppose $f : U \setminus \{P\} \to \mathbb{C}$ is holomorphic and injective so that P is an isolated singularity.

   (a) Prove that P is not an essential singularity.
   Hint: Consider $\overline{D}(P, r) \subsetneq U$ and show that $f(D(P, r) \setminus \{P\})$ is not dense in $\mathbb{C}$.

   (b) Suppose the singularity at P is removable. Prove that the holomorphic extension of f to all of U is injective.
   Hint: If we had that $\hat{f}(P) = \hat{f}(Q)$ for some $Q \neq P$, then the open mapping should imply that $f(w) = f(z)$ for some w, z near P, Q respectively.

   (c) Suppose the singularity at P is not removable. Prove that f has a simple pole.

   (d) Use the previous parts to classify all conformal maps from $\mathbb{C} \setminus \{0\}$ to itself.
   Note: This means your answer should be of the form "A holomorphic function $\phi : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ is conformal if and only if $\cdots$".

3. (a) Find a conformal map from the upper half-plane to $\{z : |\mathrm{Re}\, z| < \frac{\pi}{2}\}$.
   Hint: Consider a logarithm.

   (b) Show that $\phi(w) = -w + \sqrt{w^2 - 1}$ maps the upper half-plane conformally onto the upper half-disc $\{z : |z| < 1, \mathrm{Im}\, z > 0\}$. Which branch of the square root function should be taken here? What is the inverse of ϕ?
   Hint: Consider the image of the real axis under ϕ.

   (c) Find a conformal map from the slit unit disc $D(0, 1) \setminus \{x + i0 : 0 \le x \le 1\}$ to the strip $\{z : |\mathrm{Re}\, z| < 1\}$. You may express your answer as the composition of conformal maps.

On your own: Greene & Krantz: Ch. 6, Exercises 32, 33. Also, the following exercises:

1. Suppose that $U, V \subset \mathbb{C}$ are open and that there exists a conformal map $\phi : U \to V$. Prove that if U is holomorphically simply connected, then so is V. Conclude that the unit disc $D(0, 1)$ and the punctured disc $D(0, 1) \setminus \{0\}$ are not conformally equivalent.

2. Give a complete proof that Lemma 6.2.2, filling in any missing details from class/the text. In particular, why is it enough to know that $\phi_a(D) \subset D$ whenever $|a| < 1$ and $\phi_a \circ \phi_{-a}(z) = z$?

3. (A "Chapter 1"-type exercise to shed a little more light on Rouché's theorem) Suppose $z, w \in \mathbb{C}$ and that $|z - w| = |z| + |w|$. Prove that there exists $\omega \in \mathbb{C}$ of unit modulus such that ωz lies on the nonnegative real line and ωw lies on the nonpositive real line. This shows that z, w lie on "opposite" sides of a line through the origin.

4. Let $f(z)$ be holomorphic on the unit disc D with $|f(z)| \le 1$ and $f(\alpha) = 0$ for some $\alpha \in D$. Prove that $|f(z)| \le |\frac{z-\alpha}{1-\bar{\alpha}z}|$.

   Note: This was a recent qualifier problem. It is closely related to the Schwarz-Pick lemma, but try to limit yourself to Proposition 5.5.1 and Lemma 6.2.2.

5. Find a conformal map from the upper half plane to $\mathbb{C} \setminus (-\infty, 1]$.

Reading: Greene & Krantz, §6.1-6.3, §6.6.